Principles of Communications

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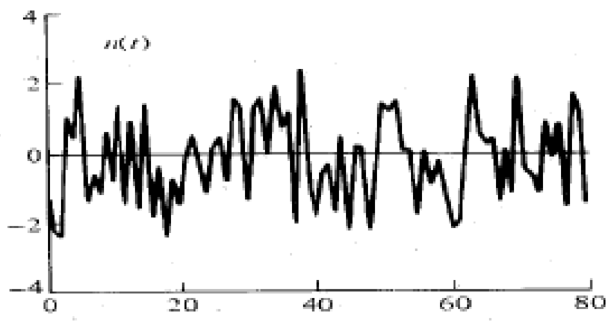
Chapter 2: Signal, Random Process, and Spectra

Textbook: 2.1-2.6, 5.1-5.3

Signal and Noise in Communication Systems

- In communication systems, the received waveform is usually categorized into the desired part containing the information and the extraneous or undesired part. The desired part is called the signal, and the undesired part is called noise.
- Noise is one of the most critical and fundamental concepts affecting communication systems
- The entire subject of communication systems is all about methods to overcome the distorting or bad effects of noise
- To do so, understanding random variables and random processes becomes quite essential



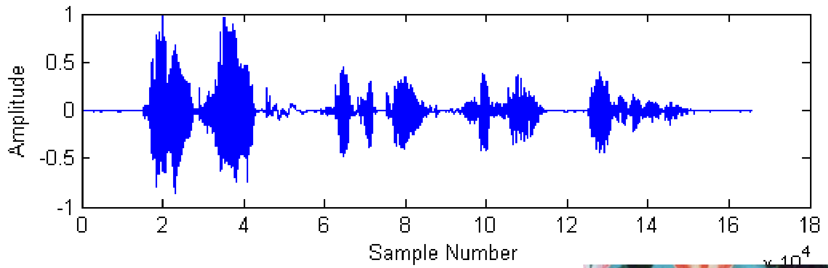
Typical noise source

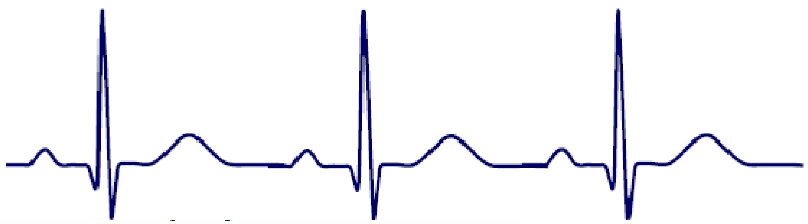
Topics to be Covered

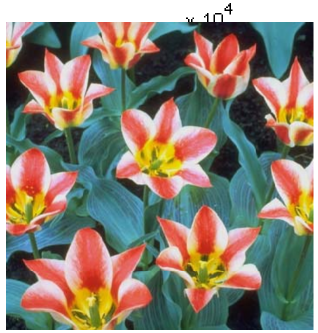
- 2.1.Signals
- 2.2.Review of probability and random variables
- 2.3.Random Processes: basic concepts
- 2.4.Guassian and White Processes

What is Signal?

- Any physical quantity that varies with time, space, or any other independent variables is called a **signal**
- In communication systems, signals are used to transmit information over a communication channel. Such signals are called **information-bearing signals**

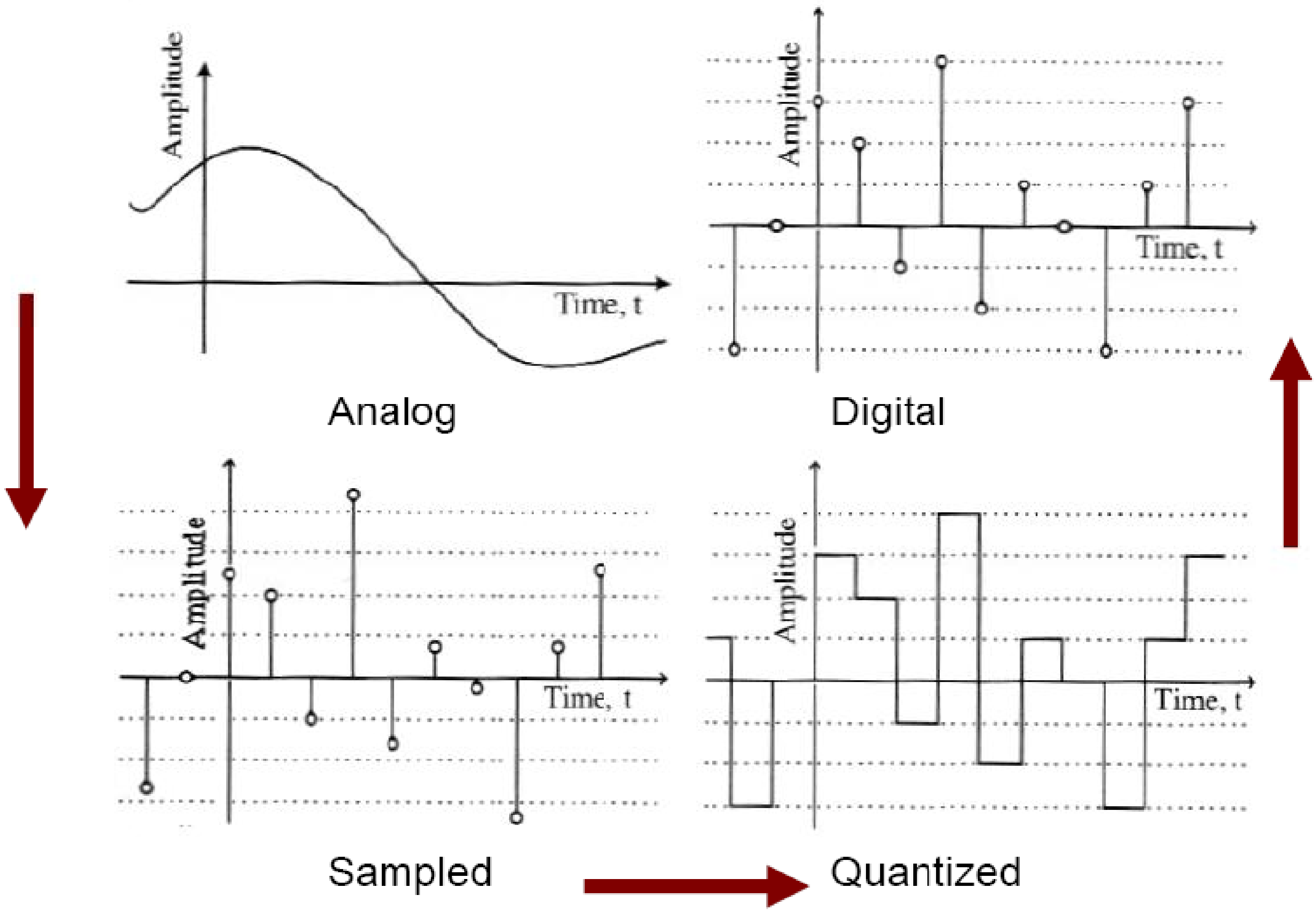




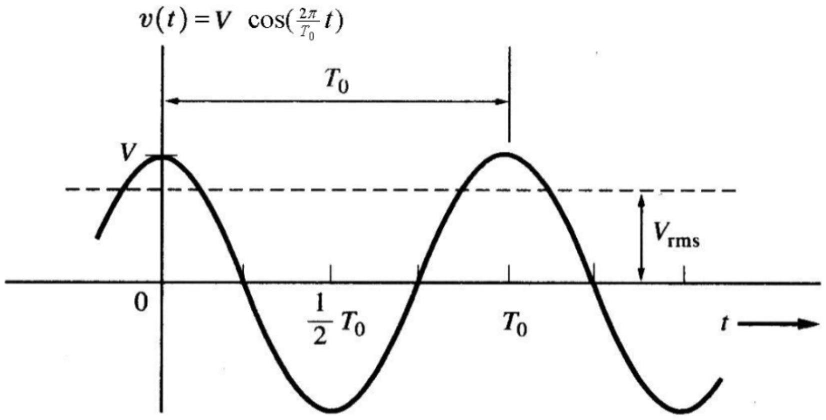


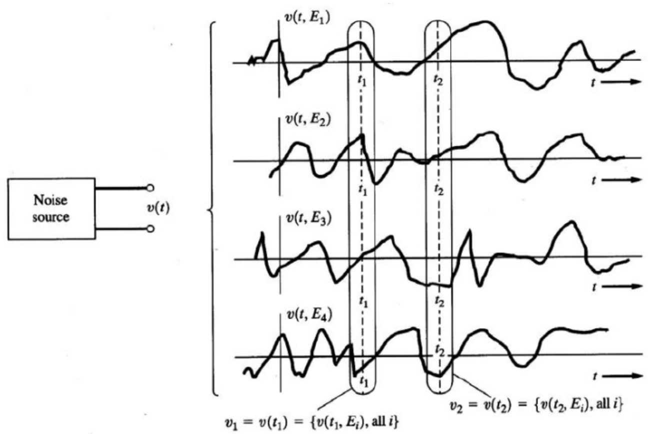
Classification of Signals

- Signals can be characterized in several ways
  - Continuous-time signal vs. discrete-time signal
  - Continuous valued signal vs. discrete-valued signal
    - o Continuous-time and continuous valued: analog signal (speech)
    - o Discrete-time and discrete valued: digital signal (CD)
    - o Discrete-time and continuous valued: sampled signal
    - o Continuous-time and discrete valued: quantized signal


Amplitude
Time, t
Analog
Digital
Sampled
Quantized

- Deterministic signal vs. random signal





Energy and Power

- Energy $$E_x = \int_{-\infty}^{\infty} \left| x(t)^2 \right| dt = \lim_{T \to \infty} \int_{-T/2}^{T/2} \left| x(t)^2 \right| dt$$
- Power $$P_x = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} \left| x(t)^2 \right| dt$$
- A signal is an **energy signal** if and only if Ex is finite
- A signal is **a** **power signal** if and only if Px is finite
- Physically realizable waveforms are of energy-type
- Mathematical models are often of power-type

Probability

- Let A be an event in a sample space S
- The probability P(A) is a real number that measures the likelihood of the event A

- Axioms of Probability

  1) $P(A) \geq 0$

  2) $P(A) \leq 1$ and $P(A) = 1$ when $A = S$

  3) Let A and B are two mutually exclusive events, i.e.

  $A \cap B = \emptyset$ Then $P(A \cup B) = P(A) + P(B)$

Elementary Properties of Probability

- $P(\bar{A}) = 1 - P(A)$

  $P(\emptyset) = 0$

- When A and B are NOT mutually exclusive

  $$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- If $A \subset B$ then $P(A) \leq P(B)$

Conditional Probability

- Consider two events A and B in a random experiment
- The probability that event A will occur GIVEN that B has occurred, P(A|B), is called the conditional probability
- The probability that both A and B occur, is called the joint probability $P(AB) = P(A \cap B)$
- Joint and conditional probabilities are related by

$$P(AB) = P(B)P(A|B) = P(A)P(B|A)$$

- Alternatively, $P(A|B) = \frac{P(AB)}{P(B)}$ $\quad P(B|A) = \frac{P(AB)}{P(A)}$
- Two events A and B are said statistically independent iff

$$P(AB) = P(A)P(B)$$

then $P(A|B) = P(A)$ and $P(B|A) = P(B)$

Law of Total Probability

- Let $A_j, \quad j = 1, 2, \ldots, n$ be mutually exclusive events with $A_i \cap A_j = \emptyset, \ \forall i \neq j$

- Then for any event B we have

$$P(B) = \sum_{j=1}^{n} P(B \cap A_j) = \sum_{j=1}^{n} P(B|A_j) P(A_j)$$

Bayes' Theorem

- An extremely useful relationship for conditional probabilities is Bayes' theorem
- Let $A_i, \quad i = 1, 2, \ldots, n$ are mutually exclusive events such that $\cup_{i=1}^{n} A_i = S$ and B is an arbitrary event with nonzero probability. Then

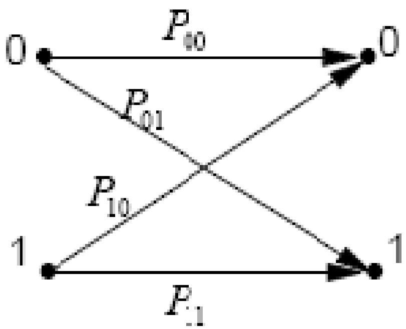
$$\begin{aligned} P(A_i|B) &= \frac{P(A_i, B)}{P(B)} \\ &= \frac{P(B|A_i)P(A_i)}{\sum_{j=1}^{n} P(B|A_j)P(A_j)} \end{aligned}$$

- This formula will be used to derive the structure of the optimal receiver

Example

- Consider a binary communication system

$P(0) = 0.3$



$P(1) = 0.7$

P_{01} = P(receive 1 | sent 0) = 0.01

P_{00} = P(receive 0 | sent 0) = 1- P_{01} = 0.09

P_{10} = P(receive 0 | sent 1) = 0.1

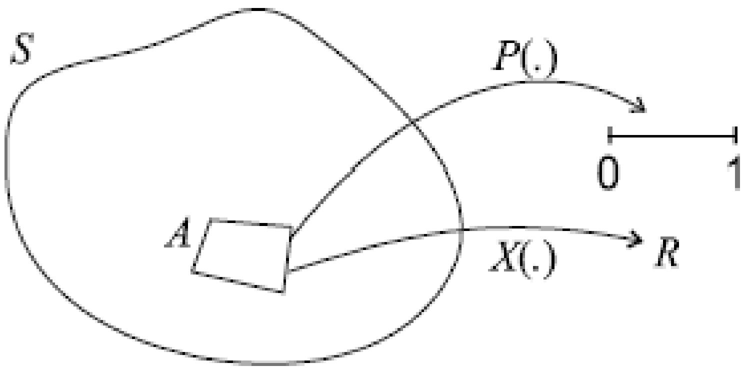
P_{11} = P(receive 1 | sent 1) = 1- P_{10} = 0.9

- What is the probability that the output of this channel is 1?
- Assuming that we have observed a 1 at the output, what is the probability that the input to the channel was a 1?

Random Variables (r.v.)

- A r.v. is a real-valued function assigned to the events of the sample space S. denoted by capital letters X, Y, etc

$$X(.) : A \subset S \to x \in R \quad X(A) = x$$



- A r.v. may be
  - Discrete-valued: range is finite (e.g. {0,1}), or countable infinite (e.g. {1,2,3 …})
  - Continuous-valued: range is uncountable infinite (e.g. $\mathcal{R}$)

- The Cumulative distribution function (CDF), or simply the probability distribution of a r.v. X, is

$$F_X(x) \stackrel{\Delta}{=} P(X \leq x)$$

- Key properties of CDF
  1. $0 \leq F_X(x) \leq 1$ with $F_X(-\infty) = 0$ and $F_X(\infty) = 1$
  2. $F_X(x)$ is a non-decreasing function of x
  3. $F(x_1 < X \leq x_2) = F_X(x_2) - F_X(x_1)$

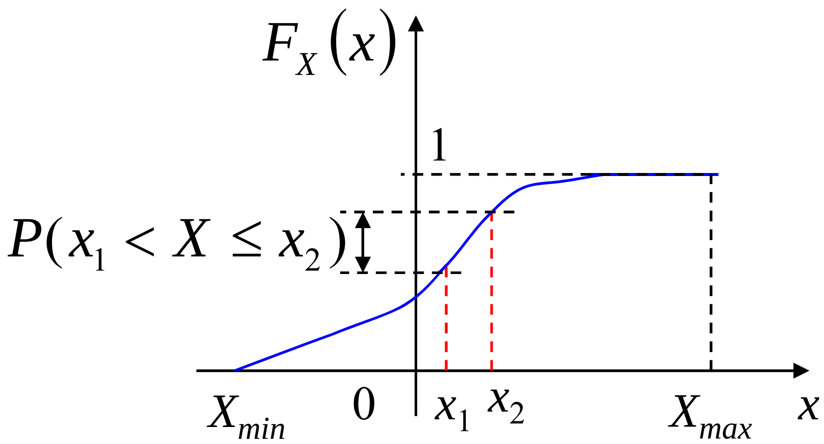
Probability Density Function (PDF)

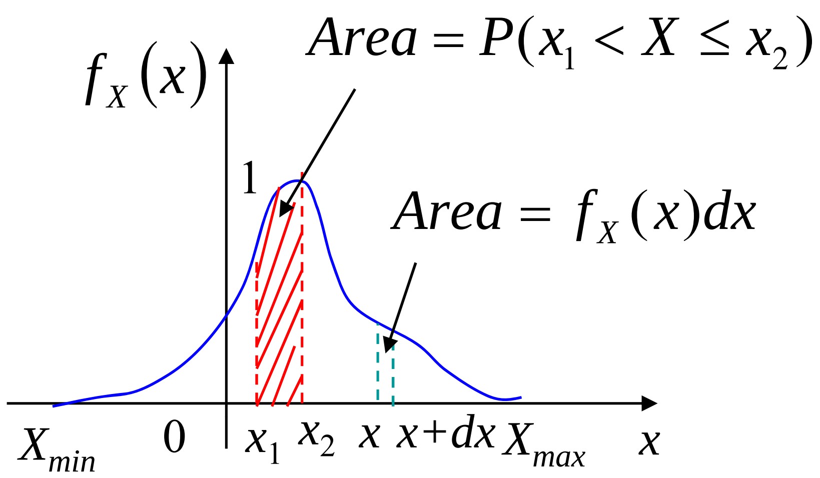
- The PDF, of a r.v. X, is defined as

$$f_X(x) \overset{\Delta}{=} \frac{d}{dx} F_X(x) \quad \text{or} \quad F_X(x) = \int_{-\infty}^{x} f_X(y)\,dy$$

- Key properties of PDF

1. $p_X(x) \geq 0$
2. $\int_{-\infty}^{\infty} p_X(x)\,dx = 1$
3. $P(x_1 < X \leq x_2) = P_X(x_2) - P_X(x_1) = \int_{x_1}^{x_2} p_X(x)\,dx$





Joint Distribution

- In many situation, one must consider TWO or more r.v.'s $\Rightarrow$ joint distribution function
- Consider 2 r.v.'s X and Y, joint distribution function is defined as $F_{XY}(x,y) = P(X \le x, Y \le y)$

  and joint PDF is $f_{XY}(x,y) = \dfrac{\partial^2 F_{XY}(x,y)}{\partial x \partial y}$
- Key properties of joint distribution

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_{XY}(x,y)dxdy = 1$$

$$P(x_1 < X \le x_2, y_1 < Y \le y_2) = \int_{y_1}^{y_2}\int_{x_1}^{x_2} p_{XY}(x,y)dxdy$$

- Marginal distribution

$$P_X(x) = P(X \le x, -\infty < Y < \infty) = \int_{-\infty}^{\infty}\int_{-\infty}^{x} p_{XY}(\alpha,\beta)\, d\alpha\, d\beta$$

$$P_Y(y) = \int_{-\infty}^{y}\int_{-\infty}^{\infty} p_{XY}(\alpha,\beta)\, d\alpha\, d\beta$$

- Marginal density

$$p_X(x) = \int_{-\infty}^{\infty} p_{XY}(x,\beta)\, d\beta$$

- X and Y are said to be *independent* iff

$$P_{XY}(x,y) = P_X(x)P_Y(y)$$

$$p_{XY}(x,y) = p_X(x)p_Y(y)$$

Statistical Averages

- Let X be a r.v. Then X can either continuous or discrete. For the moment, consider a discrete r.v. which takes on the possible values $x_1, x_2, \ldots, x_M$ with respective probabilities $P_1, P_2, \ldots, P_M$. Then the mean or expected value of X is

$$m_X = E[X] = \sum_{i=1}^{M} x_i P_i$$

where "E[]" denotes the expectation operation (statistically averaging)

- If X is continuous, then

$$m_X = E[X] = \int_{-\infty}^{\infty} x f_X(x)dx$$

- This is the first moment of the random variable X
- Let g(X) be a function of X, then

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) p_X(x)dx$$

- For the special case of g(X) =x^n, we obtain the *n^{th} moment* of X, that is

$$E[X^n] = \int_{-\infty}^{\infty} x^n p_X(x)dx$$

- Let n = 2, we have the mean-square value of X as

$$E[X^2] = \int_{-\infty}^{\infty} x^2 p_X(x)dx$$

- *n*-th Central moment is

$$E\left[(X-m_X)^n\right]=\int_{-\infty}^{\infty}(x-m_x)^n f_X(x)dx$$

- The expected value of second central moment (n=2) is called *variance*

$$\begin{aligned}\sigma_X^2 &= E\left[(X-m_X)^2\right] \\ &= E\left[X^2-2m_X X+m_X^2\right] \\ &= E\left[X^2\right]-m_X^2\end{aligned}$$

- σ_X, square-root of the variance, is called the *standard deviation*. It is the average distance from the mean, a measure of the concentration of X around the mean

Correlation

- In considering multiple varaibles, the joint moments like correlation and covariance between pairs of r.v.s are most useful
- Correlation of the two r.v. X and Y is defined as

$$R_{XY} = E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_{XY}(x,y)dxdy$$

- Correlation of X and Y is the mean of the product X and Y
- Correlation of the two centered r.v. X-E[X] and Y-E[Y], is called the covariance of X and Y

$$Cov_{XY} = E[(X - E[x])(Y - E[y])]$$

- The covariance of X and Y normalized w.r.t. $\sigma_X \sigma_Y$ is referred to the correlation coefficient of X and Y:

$$\rho_{XY} = \frac{Cov(XY)}{\sigma_X \sigma_Y}$$

- X and Y are uncorrelated iff their correlation coefficient is 0

$$\rho_{XY} = 0 \quad \Rightarrow E[XY] = E[X]E[Y]$$

- X and Y are orthogonal iff their correlation is 0

$$R_{XY} = E[XY] = 0$$

- If X and Y are independent, then they are uncorrelated. However, the converse is not true (The Gaussian case is the only exception)

Some Useful Probability Distributions

- Discrete Distribution
  - Binary distribution
  - Binomial distribution
- Continuous Distribution
  - Uniform distribution
  - Gaussian distribution (most important one)
  - Rayleigh distribution (very important in mobile and wireless communications)

Binary Distribution

- Let X be a discrete random variable that has two possible values, say X = 1 or X = 0. Distribution of X can be described by probability mass function (pmf)

$$\begin{aligned} p(x) &= P(X=x) \\ &= \begin{cases} 1-p & x=0 \\ p & x=1 \end{cases} \end{aligned}$$

- This is frequently used to model binary data
- Mean:

$$m_X = \sum_x x \cdot p(x) = 0 \cdot (1-p) + 1 \cdot p = p$$

- Variance

$$\begin{aligned} \sigma_X^2 &= \sum_x (x - m_X)^2 \cdot p(x) \\ &= (-p)^2 \cdot (1-p) + (1-p)^2 \cdot p = p(1-p) \end{aligned}$$

Binomial Distribution

- Let $Y = \sum_{i=1}^{n} X_i$ where $\{X_i, i = 1, \ldots, n\}$ are independent binary r.v.s with

$$p_X(x) = \begin{cases} 1-p & x=0 \\ p & x=1 \end{cases}$$

- Then

$$p_Y(k) = \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k} \quad \text{where} \quad \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

- That is, the probability that $Y = k$ is the probability that k of the X_i are equal to 1 and n-k are equal to 0
- Mean: $m_Y = np$
- Variance: $np(1-p)$

Example

- Suppose that we transmit a 31-bit long sequence with error correction capability up to 3 bit errors
- If the probability of a bit error is p = 0.001, what is the probability that this sequence is received in errors?

$$P(\text{sequence error}) = 1 - P(\text{correct sequence})$$

$$= 1 - \sum_{i=0}^{3} \binom{31}{i} (0.001)^i (0.999)^{31-i} \approx 3 \cdot 10^{-8}$$

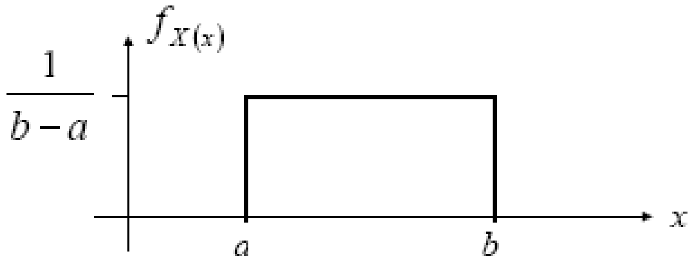
- If no error correction is used, the error probability is

$$= 1 - (0.999)^{31} \approx 0.0305$$

Uniform Distribution

- The pdf of uniform distribution is given by

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$



$$E[X] = \frac{a+b}{2}$$

$$\sigma_X^2 = \frac{(a-b)^2}{12}$$

Any example?

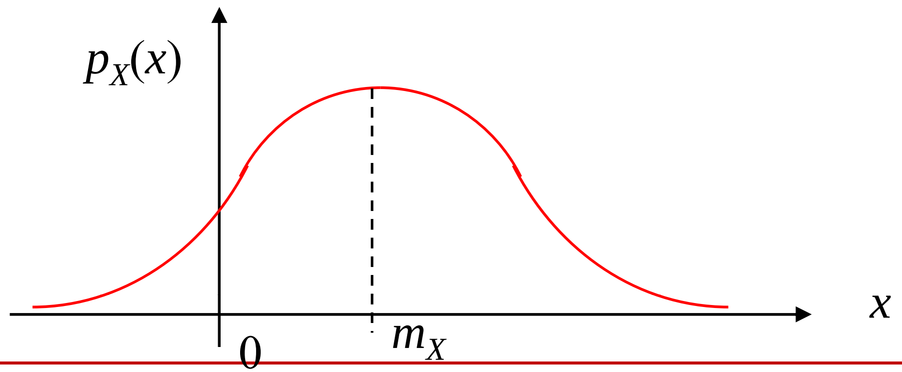
Gaussian Distribution

- The Gaussian distribution, also called normal distribution, is by far the most important distribution in the statistical analysis of communication systems
- The PDF of a Gaussian r.v. is

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} \exp\left[-\frac{1}{2\sigma_X^2}(x - m_X)^2\right]$$

- A Gaussian r.v. is completely determined by its mean and variance, and hence usually denoted as

$$X \sim N\left(m_X, \sigma_X^2\right)$$

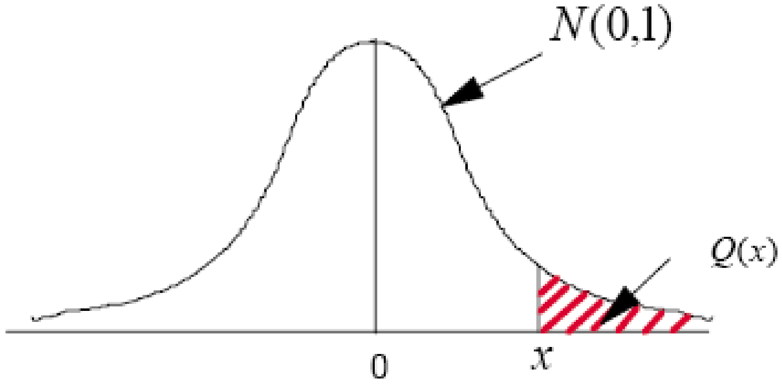


The Q-Function

- The Q-function is a standard form to express error probabilities without a closed form

$$Q(x) = \int_x^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2} \right) du$$

- The Q-function is the area under the tail of a Gaussian pdf with mean zero and variance one



- Extremely important in error probability analysis!!!

More about Q-Function

- Q-function is monotonically decreasing
- Some features

$$Q(-\infty) = 1 \qquad Q(0) = \frac{1}{2} \qquad Q(\infty) = 0 \qquad Q(-x) = 1 - Q(x)$$

- Craig's alternative form of Q-function (IEEE MILCOM'91)

$$Q(x) = \frac{1}{\pi}\int_0^{\pi/2} \exp\left(-\frac{x^2}{2\sin^2\theta}\right)d\theta, \quad x \geq 0$$

- Upper bound $Q(x) \leq \frac{1}{2}e^{-x^2/2}$
- If we have a Gaussian variable $X \sim N(\mu, \sigma^2)$, then

$$\Pr(X > x) = Q\left(\frac{x-\mu}{\sigma}\right)$$

Joint Gaussian Random Variables

- $X_1, X_2, \ldots, X_n$ are jointly Gaussian iff

$$f(x_1, x_2, \ldots, x_n) = \frac{1}{(2\pi)^{n/2}(\det(\mathbf{C}))^{1/2}} \exp\left[-\frac{(\mathbf{x}-\mathbf{m})^T \mathbf{C}^{-1}(\mathbf{x}-\mathbf{m})}{2}\right]$$

  - x is a column vector $\mathbf{x} = (x_1, \ldots, x_n)^T$
  - **m** is the vector of the means $\mathbf{m} = (m_1, \ldots, m_n)^T$
  - **C** is the $n \times n$ covariance matrix

$$\mathbf{C} = \left[C_{i,j}\right] \qquad C_{i,j} = E\left[(X_i - m_i)(X_j - m_j)\right]$$

Two-Variate Gaussian PDF

- Given two r.v.s: X_1 and X_2 that are joint Gaussian

$$\mathbf{C} = \begin{bmatrix} E[(X_1 - m_1)^2] & E[(X_1 - m_1)(X_2 - m_2)] \\ E[(X_1 - m_1)(X_2 - m_2)] & E[(X_2 - m_2)^2] \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$$

- Then

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{(x_1 - m_1)^2}{\sigma_1^2} - \frac{2\rho(x_1 - m_1)(x_2 - m_2)}{\sigma_1\sigma_2} + \frac{(x_2 - m_2)^2}{\sigma_2^2}\right]\right\}$$

- For uncorrelated X and Y, i.e. $\rho = 0$

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2} \exp\left\{-\frac{1}{2}\left[\frac{(x_1 - m_1)^2}{\sigma_1^2} + \frac{(x_2 - m_2)^2}{\sigma_2^2}\right]\right\}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_1} e^{-(x_1 - m_1)^2/2\sigma_1^2} \cdot \frac{1}{\sqrt{2\pi}\sigma_2} e^{-(x_2 - m_2)^2/2\sigma_2^2}$$

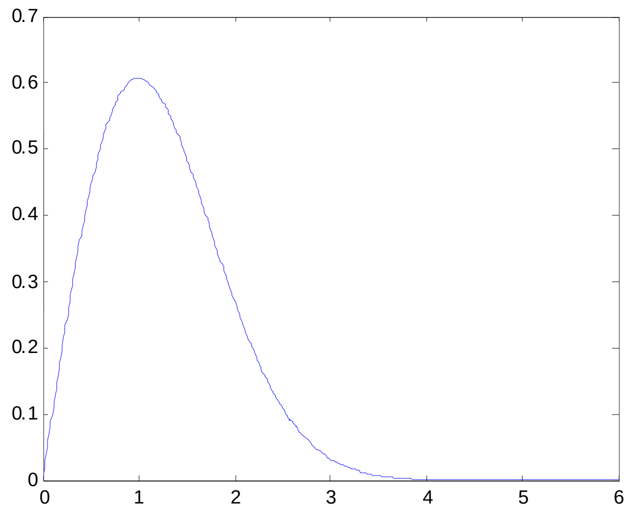
$$= f(x_1) f(x_2)$$

⇨ X1 and X2 are also independent

If X_1 and X_2 are Gaussian and uncorrelated, then they are independent.

Rayleigh Distribution

$$f_X(x) = \begin{cases} \frac{x}{\alpha^2}\exp\left(-\frac{x^2}{2\alpha^2}\right) & x \geq 0 \\ 0 & x < 0 \end{cases}$$



- Rayleigh distributions are frequently used to model **fading** for non-line of sight (NLOS) signal transmission
- Very important for mobile and wireless communications

Sums of Random Variables

- Consider a sequence of r.v.'s $\{X_1, X_2, \ldots, X_n\}$
- Weak Law of Large Numbers
  - Let $$Y = \frac{1}{n}\sum_{i=1}^{n} X_i$$
  - And assume that X_i's are uncorrelated with the same mean m_X and variance $\sigma_X^2 < \infty$
  - Then
$$\lim_{n\to\infty} P\left(\left|Y - m_X\right| \geq \varepsilon\right) = 0 \qquad \forall \varepsilon > 0$$

  So what?

i.e. the average converges to the expected value

Central Limit Theorem

- Let $\{X_1, X_2, \ldots, X_n\}$ be a set of independent random variables with common mean m_X and common variance σ_X^2
- Next let $Y = \sum_{i=1}^{n} X_i$
- Then as $n \to \infty$, the distribution of Y will tend towards a Gaussian distribution

Key Conclusion: the sum of random variables is "Gaussian"

- Thermal noise results from the random movement of many electrons – it is well modeled by a Gaussian distribution.

Example

$y_1 = \sin(x_1)$

$y_2 = \sin(x_1)+\sin(x_2)$

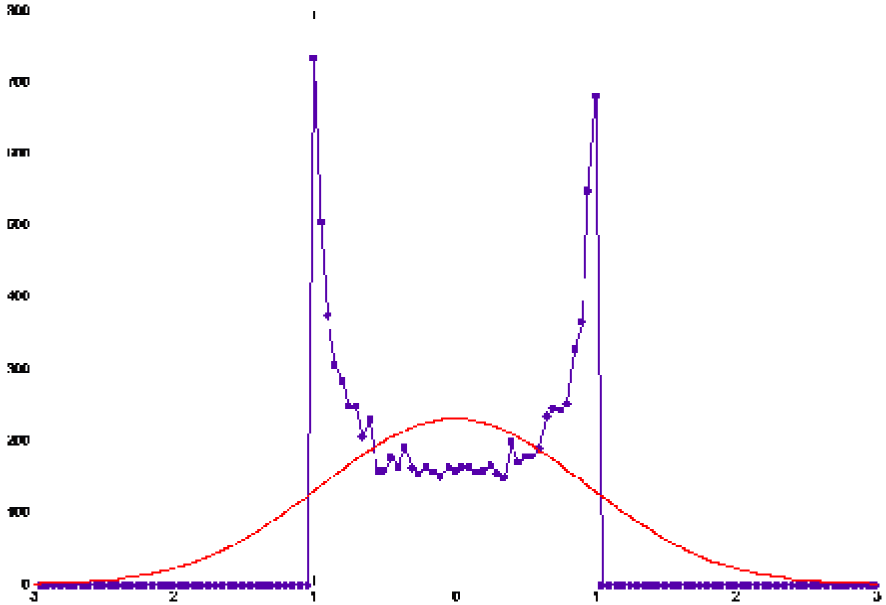
$y_4 = \sin(x_1)+\sin(x_2)+ \ldots + \sin(x_4)$

$y_8 = \sin(x_1)+\sin(x_2)+ \ldots + \sin(x_8)$

$y_{16} = \sin(x_1)+\sin(x_2)+ \ldots + \sin(x_{16})$

$x_1, x_2, \ldots, x_{16}$ are independent and uniformly distributed over 0 to 2π.

N = 1



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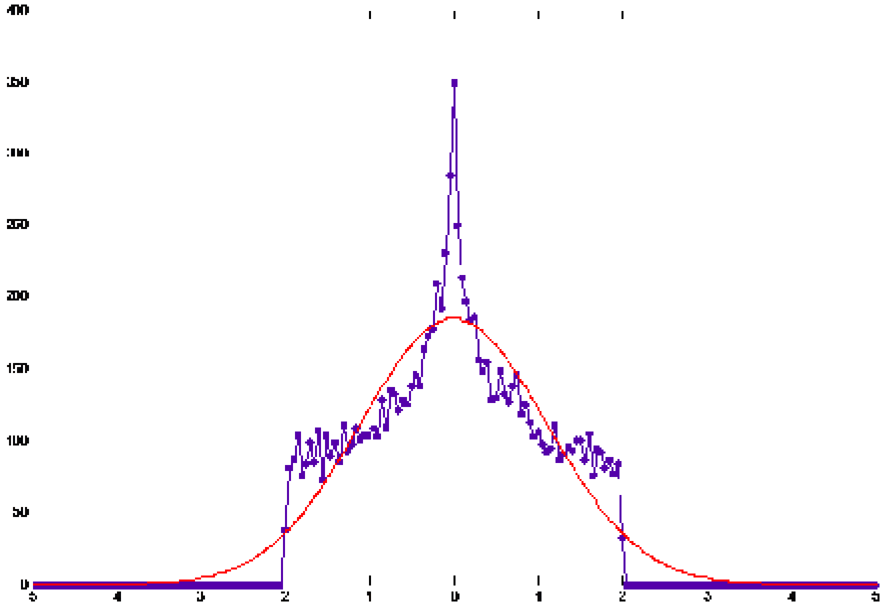
N = 2



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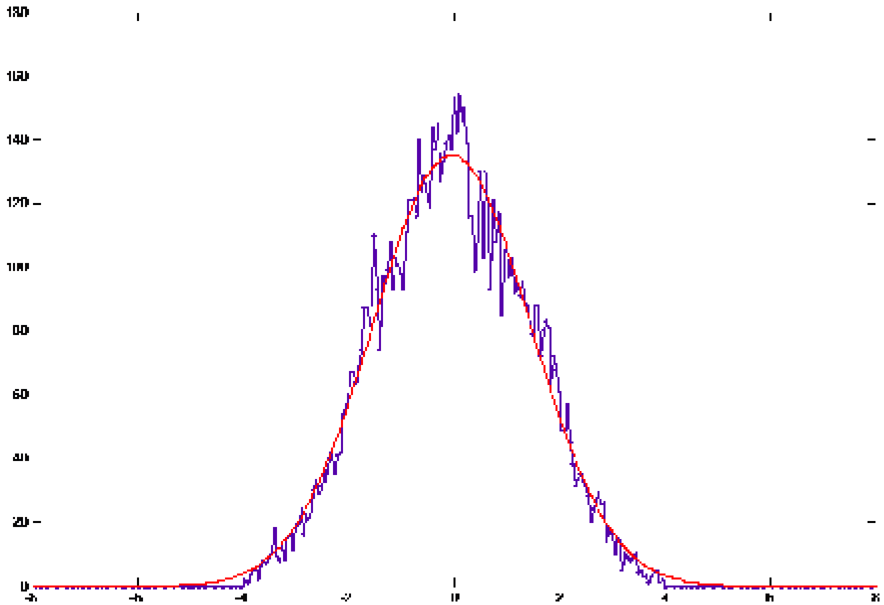
N = 4



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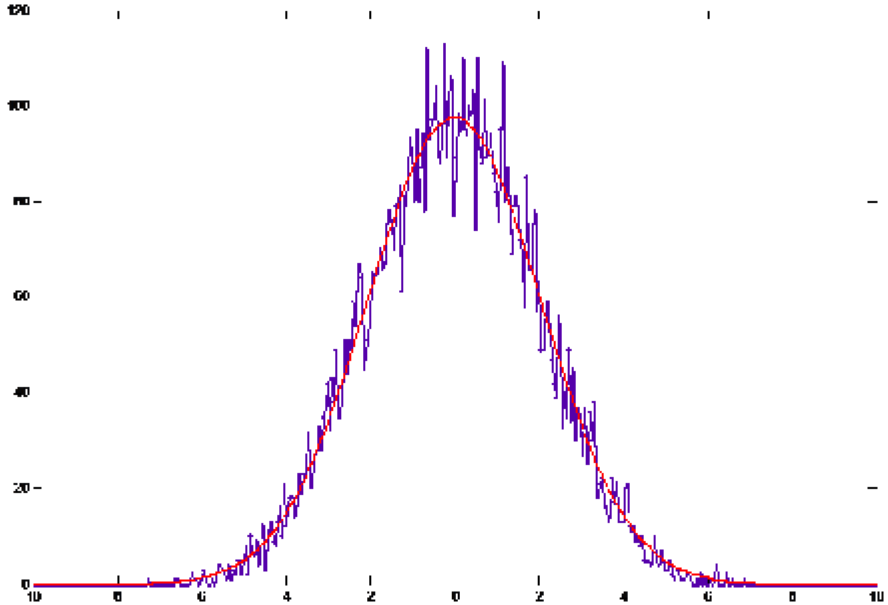
N = 8



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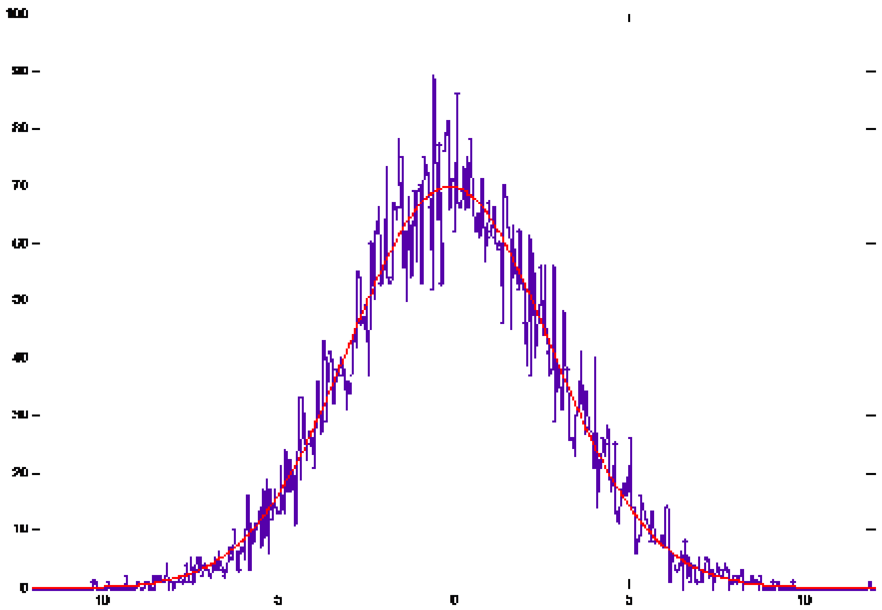
N = 16



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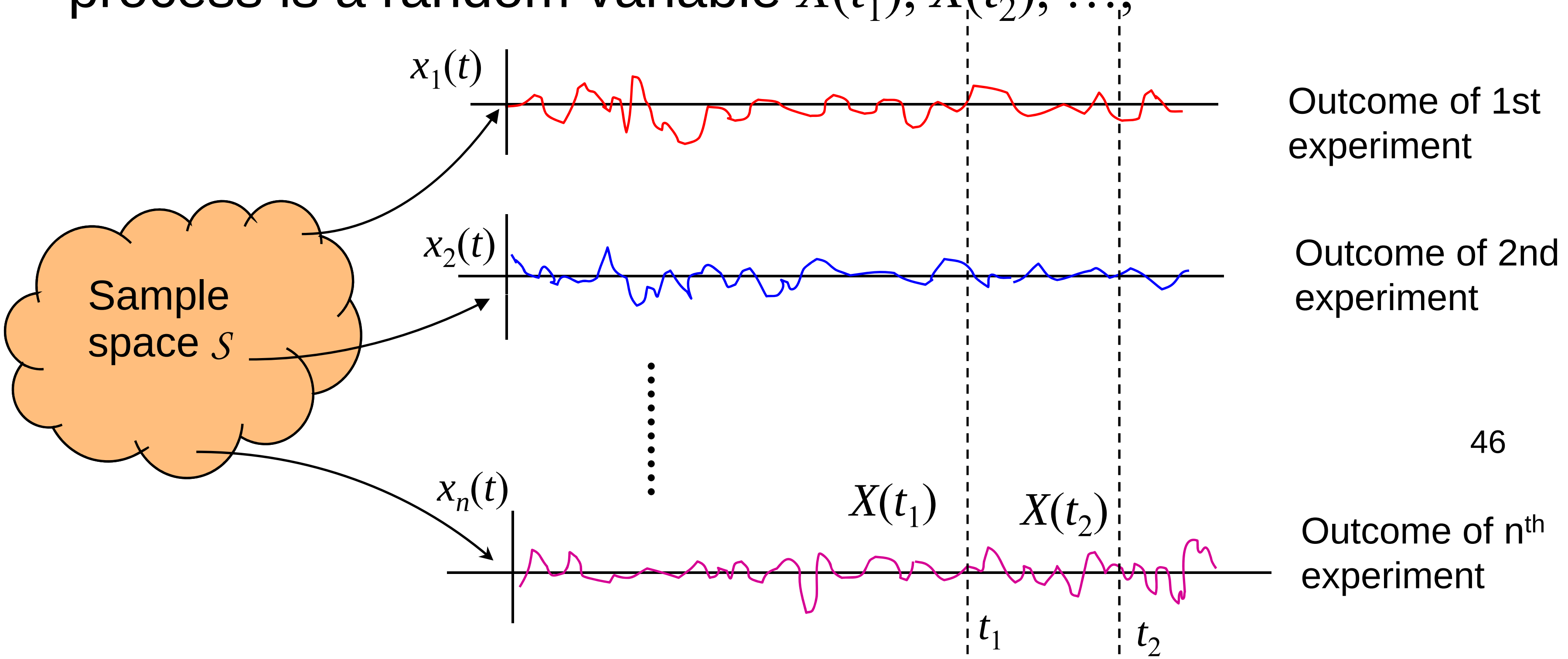
Random Process

- A random process is the natural extension of random variables when dealing with signals

- Also referred to as stochastic process or random signal

- Voices signals, TV signals, thermal noise generated by a radio receiver are all examples of random signals.

- A random process can be described as $X(t)$
- For each experiment *n*, there exists a time-function $x_n(t)$, called a sample function or realization of the random process
- At any time instant $t_1, t_2, \ldots$, the value of the random process is a random variable $X(t_1), X(t_2), \ldots,$



Statistics of Random Processes

- By sampling the random process at any time, we get a random variable
- From this view point, we can think of a random process as an infinite collection of random variables specified at time *t*: $\{X(t_1), X(t_2), \ldots, X(t_n)\}$
- Thus, a random process can be completely defined statistically as a collection of random variables indexed by time with properties defined by a joint PDF

- A random process X(t) is described by its M-th order statistics if for all $n \leq M$ and all $\{t_1, t_2, \ldots, t_n\}$ the joint pdf of $\{X(t_1), X(t_2), \ldots, X(t_n)\}$ is given
- This joint pdf is written as

$$f\left(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n\right)$$

- In order to completely specify a random process, one must given $f\left(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n\right)$ for all possible values of $\{x_i\}$ $\{t_i\}$, and for all n. This is obviously quite difficult in general

First Order Statistics on Random Processes

- The first order statistics is simply the PDF of a random variable at one particular time
- $f(x;t)$ = first order density of X(t)
- $F(x;t) = P(X(t) \leq x)$, first order distribution of $X(t)$
- **Mean** $E[X(t_0)] = E[X(t=t_0)] = \int_{-\infty}^{\infty} x f_X(x;t_0) = \overline{X}(t_0)$
- **Variance** $E\left[\left|X(t_0) - \overline{X}(t_0)\right|^2\right] = \sigma_X^2(t_0)$

Second-Order Statistics on Random Processes

- Second-order statistics means the joint PDF of $X(t_1)$ and $X(t_2)$ for all choices t_1 and t_2.
- Auto-correlation function: Let $t_1 = t$, and $t_2 = t+\tau$,

$$R_X(t;\tau) = E\left[X(t)X(t+\tau)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f(x_1, x_2; t, t+\tau) dx_1 dx_2$$

- The physical meaning of $R_X(t;\tau)$ is a measure of the relationship of the function $X(t)$ and $X(t+\tau)$ (correlation within a process)
- In general, the autocorrelation function is a function of both *t* and τ.

Example

- Given a stochastic process $X(t) = A\cos(2\pi ft + \theta)$, where θ is a random variable uniformly distributed from $-\pi$ to π
- At each *t*, X(t) can be viewed as a function of θ
- The mean is $E[X(t)] = \int_{-\pi}^{\pi} A\cos(2\pi ft + \theta)\frac{1}{2\pi}d\theta = 0$
- The auto-correlation is

$$\begin{aligned} E[X(t)X(t+\tau)] &= E\left[A\cos(2\pi ft + \theta)A\cos(2\pi f(t+\tau) + \theta)\right] \\ &= \frac{A^2}{2}E\left[\cos(4\pi ft + 2\pi f\tau + 2\theta) + \cos(2\pi f\tau)\right] \\ &= \frac{A^2}{2}\int_{-\pi}^{\pi}\frac{1}{2\pi}\cos(4\pi ft + 2\pi f\tau + 2\theta)d\theta + \frac{A^2}{2}\int_{-\pi}^{\pi}\frac{1}{2\pi}\cos(2\pi f\tau)d\theta \\ &= 0 + \frac{A^2}{2}\cos(2\pi f\tau) \end{aligned}$$

$$\Rightarrow R_X(t;\tau) = \frac{A^2}{2}\cos(2\pi f\tau)$$

Stationary Processes

- A stochastic process is said to be stationary if for any n and τ the following holds:

$$f_X(x_1, x_2, \cdots x_n; t_1, t_2, \cdots t_n) = f_X(x_1, x_2, \cdots x_n; t_1 + \tau, t_2 + \tau, \cdots t_n + \tau) \quad (1)$$

- Therefore,
  - Te first-order statistics is independent of t

$\Longrightarrow$ mean $$E\{X(t)\} = \int_{-\infty}^{\infty} x f_X(x) dx = m_X \quad (2)$$

  - The second-order statistics only depends on the gap between t1 and t2

$\Longrightarrow$ Autocorrelation function

$$R_X(t_1, t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_X(x_1, x_2, t_2 - t_1) dx_1 dx_2$$
$$= R_X(t_2 - t_1) = R_X(\tau), \quad \text{where } \tau = t_2 - t_1 \quad (3)$$

Wide-Sense Stationary

- Our engineers often care about the first- and second-order statistics only
- A random process is said to be WSS when conditions (2) and (3) hold
- A random process is said to be strictly stationary when condition (1) holds
- Example:

$$X(t) = A\cos(2\pi f t + \theta) \text{ , where } \theta \sim U(-\pi, \pi)$$

$$R_X(t_1, t_2) = \frac{A^2}{2}\cos(2\pi f \tau)$$

Only depends on the time difference $\tau = t_2 - t_1$

⇒ Thus, X(t) is WSS

Averages and Ergodic

- Ensemble averaging

$$\overline{X}(t) \stackrel{\Delta}{=} E\left[X(t)\right] = \int_{-\infty}^{\infty} x p(x;t) dx$$

$$R_X(t_1, t_2) \stackrel{\Delta}{=} E\left[X(t_1) X(t_2)\right] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_2 p(x_1, x_2; t_1, t_2) dx_1 dx_2$$

- Time averaging

$$< X(t) > \stackrel{\Delta}{=} \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t) dt$$

$$< X(t) X(t-\tau) > \stackrel{\Delta}{=} \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t) x(t-\tau) dt$$

- In general, ensemble averages and time averages are not equal
- A r.p. X(t) is said to be Ergodic if all time averages and Ensemble averages are equal

Random Processes in the Frequency Domain: Power Spectral Density

- Let X(t) denote a random process and let $x(t,n)$ denote a sample function of this process
- Truncate the signal by defining

$$x_T(t,n) = \begin{cases} x(t,n) & |t| < T/2 \\ 0 & \text{otherwise} \end{cases}$$

  in order to get an energy signal.

- Performing a Fourier transform on $x_T(t,n)$, we get $X_T(f,n)$
- According to Parseval theorem

$$\int_{-\infty}^{\infty} x_T^2(t,n)dt = \int_{-\infty}^{\infty} |X_T(f,n)|^2 df$$

$|X_T(f,n)|^2$: energy spectral density

- Then the power spectral density is the average energy spectral density per time unit, i.e.

$$\frac{|X_T(f,n)|^2}{T}$$

- Letting $T \to \infty$, we define the power spectral density for the sample function:

$$S_X(f,n) = \lim_{T\to\infty} \frac{|X_T(f,n)|^2}{T}$$

- If we take the ensemble average, the power spectral density (PSD) of the random process is

$$S_X(f) = E\left[\lim_{T\to\infty} \frac{|X_T(f,n)|^2}{T}\right] = \lim_{T\to\infty} \frac{E[|X_T(f,n)|^2]}{T} \quad (4)$$

Watts/Hz

⇨ *The general definition of power spectral density*

PSD of Stationary Process

Wiener-Khinchin theorem

For a stationary random process X(t), the PSD is equal to the Fourier Transform of the autocorrelation function, i.e.,

$$S_X(f) \leftrightarrow R_X(\tau) \quad \begin{cases} R_X(\tau) = \int_{-\infty}^{\infty} S_X(f) \exp(j2\pi f\tau)\, df \\ S_X(f) = \int_{-\infty}^{\infty} R_X(\tau) \exp(-j2\pi f\tau)\, d\tau \end{cases}$$

- In general, $S_X(f)$ is a measure of the relative power in the random signal at each frequency component

$$R_X(0) = \int_{-\infty}^{\infty} S_X(f)\, df = \text{total power}$$

Gaussian Process

- The importance of Gaussian processes in communication systems is due to that thermal noise in electronic devices can be closely modeled by a Gaussian process
- **Definition**:
  - A random process X(t) is a Gaussian process if for all n and all (t1, t2, …, tn), the random variables {X(t1), X(t2), …, X(tn)} have a joint Gaussian density function

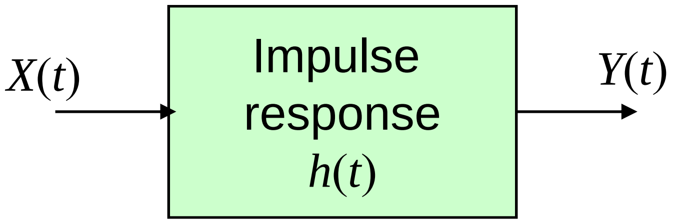
$$f(x_1, x_2, \ldots, x_n) = \frac{1}{(2\pi)^{n/2}(\det(\mathbf{C}))^{1/2}} \exp\left[-\frac{(\mathbf{x}-\mathbf{m})^T \mathbf{C}^{-1} (\mathbf{x}-\mathbf{m})}{2}\right]$$

Properties of Gaussian Processes

- If a Gaussian random process is wide-sense stationary, then it is also stationary
- Any sample point from a Gaussian random process is a Gaussian random variable
- If the input to a linear system is a Gaussian random process, then the output is also a Gaussian random process.

Random Process Transmission Through Linear Systems

- Consider a linear system



$$Y(t) = X(t) * h(t) = \int_{-\infty}^{\infty} h(\tau) X(t-\tau) d\tau$$

- The mean of the output random process Y(t)

$$\overline{Y}(t) = E[Y(t)] = \int_{-\infty}^{\infty} h(\tau) E[X(t-\tau)] d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau) \overline{X}(t-\tau) d\tau$$

If X(t) is WSS

$$= \overline{X} \int_{-\infty}^{\infty} h(\tau) d\tau = \overline{X} \cdot H(0)$$

where H(0) is the zero-frequency response of the system

- The autocorrelation of Y(t)

$$R_Y(t,u) = E[Y(t)Y(u)]$$
$$= E\left[\int_{-\infty}^{\infty} h(\tau_1)X(t-\tau_1)d\tau_1 \int_{-\infty}^{\infty} h(\tau_2)X(u-\tau_2)d\tau_2\right]$$
$$= \int_{-\infty}^{\infty} h(\tau_1)d\tau_1 \int h(\tau_2)E[X(t-\tau_1)X(u-\tau_2)]d\tau_2$$

If X(t) is WSS

$$R_Y(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)R_X(\tau-\tau_1+\tau_2)d\tau_1 d\tau_2$$

If input is a WSS random process, the output is also a WSS random process

Relation Among the Input-Output PSD

- Autocorrelation of Y(t)

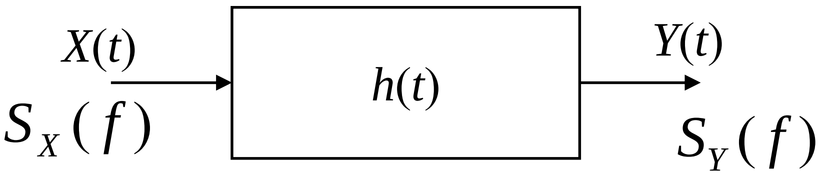
$$R_Y(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)R_X(\tau-\tau_1+\tau_2)d\tau_1 d\tau_2$$

$$= \int_{-\infty}^{\infty} h(\tau_2)d\tau_2 \int_{-\infty}^{\infty} h(\tau_1)R_X(\tau+\tau_2-\tau_1)d\tau_1$$

$$= \int_{-\infty}^{\infty} h(\tau_2)\left[h(\tau+\tau_2) * R_X(\tau+\tau_2)\right]d\tau_2$$

$$= h(-\tau) * h(\tau) * R_X(\tau)$$

- PSD of Y(t): $S_Y(f) = |H(f)|^2 S_X(f)$

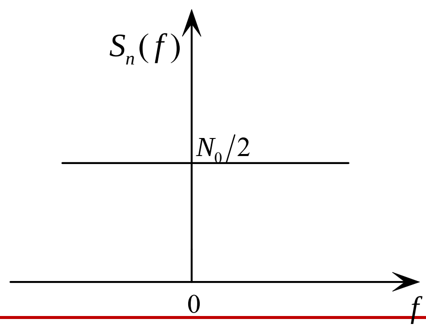


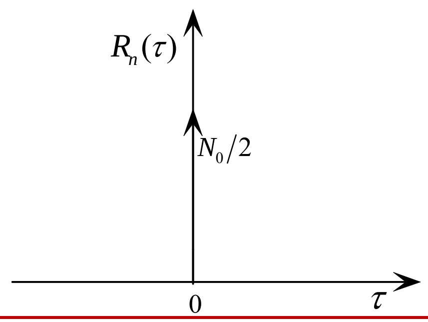
$$S_Y(f) = |H(f)|^2 S_X(f)$$ ← **Key Results**

Noise

- Noise is a critical component in the analysis of the performance of communication receivers
- Often assumed to be Gaussian and stationary
- The mean is taken to be zero while the autocorrelation is usually specified by the power spectral density
- The noise is a white noise, when all frequency components appear with equal power (white is used in white light for a similar reason)

$$S_n(f) = N_0/2 \qquad\qquad R_n(\tau) = \frac{N_0}{2}\delta(\tau)$$

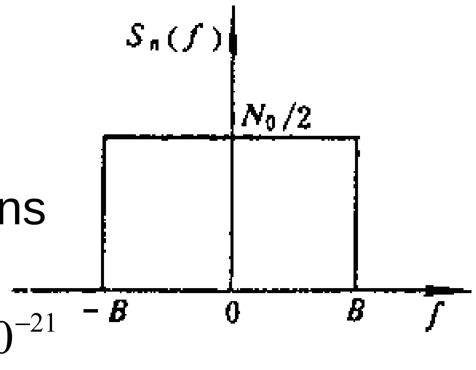




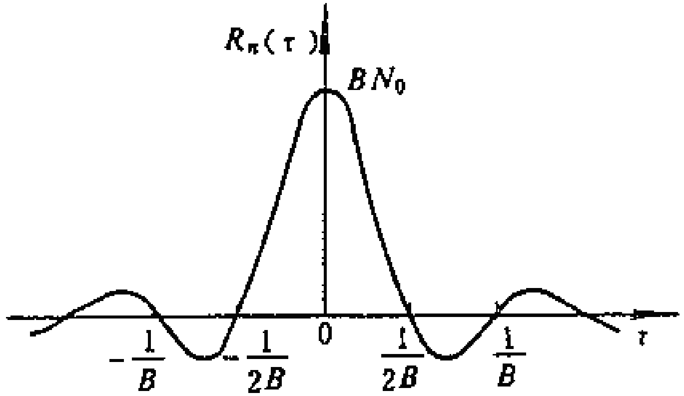
White noise is completely uncorrelated!

Bandlimited Noise

White noise → Filter Bandwidth B Hz → Bandlimited white noise $n(t)$



(a)



(b)

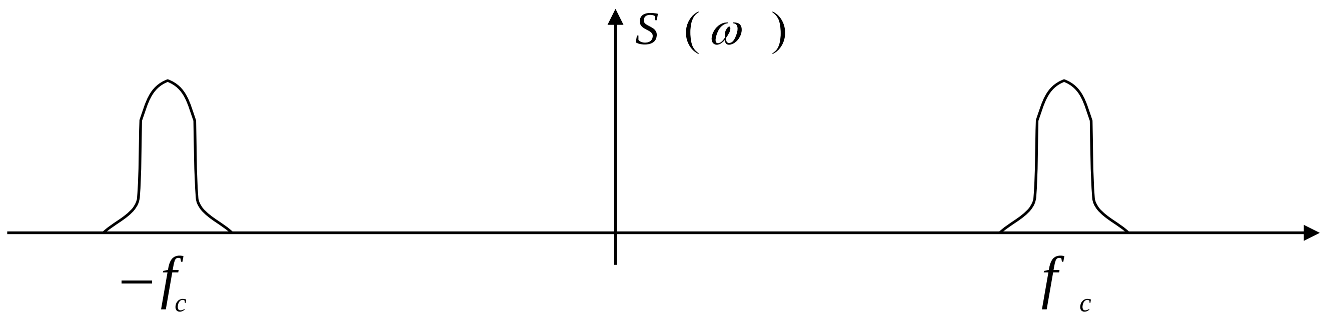
In most applications

$$N_0 = KT = 4.14\times10^{-21}$$
$$= -174 \text{ dBm/Hz}$$

At what sampling rate to sample the noise can we get uncorrelated realizations?

Narrow-Band Random Process

- The bandwidth of the signal is limited to a narrow band around a central frequency fc >> 0



- Canonical form of a narrow band process

$$X(t) = X_I(t)\cos(2\pi f_0 t) - X_Q(t)\sin(2\pi f_0 t)$$

In-phase component ($X_I(t)$) Quadrature component ($X_Q(t)$)

Narrow band Noise

- Let n(t) be a zero-mean, stationary noise

$$n(t) = n_c(t)\cos\omega_0 t - n_s(t)\sin\omega_0 t$$

- Find the statistics of $n_c(t)$ and $n_s(t)$
- Result 1：

$$\boxed{E\{n(t)\} = E\{n_c(t)\} = E\{n_s(t)\} = 0}$$

- Proof:

$$E[n(t)] = E[n_c(t)]\cos\omega_0 t - E[n_s(t)]\sin\omega_0 t$$

  - Since n(t) is stationary， zero-mean, for any t, we have

$$E[n(t)] = 0$$

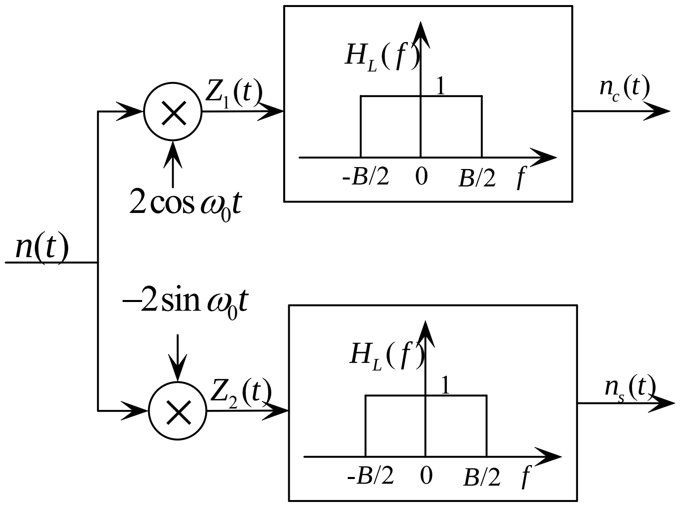
  - Thus：

$$E\{n_c(t)\} = E\{n_s(t)\} = 0$$

- Result 2:

$$S_{n_c}(f) = S_{n_s}(f) = \begin{cases} S_n(f - f_0) + S_n(f + f_0), & |f| \le B/2 \\ 0 & \text{otherwise} \end{cases}$$

- Proof



- Result 3：for the same t, $n_c(t)$ and $n_s(t)$ are uncorrelated or independent

$$R_{n_c n_s}(0) = 0$$

- Result 4：

$$E\left\{n^2(t)\right\} = E\left\{n_c^{\ 2}(t)\right\} = E\left\{n_s^{\ 2}(t)\right\} = \sigma^2$$

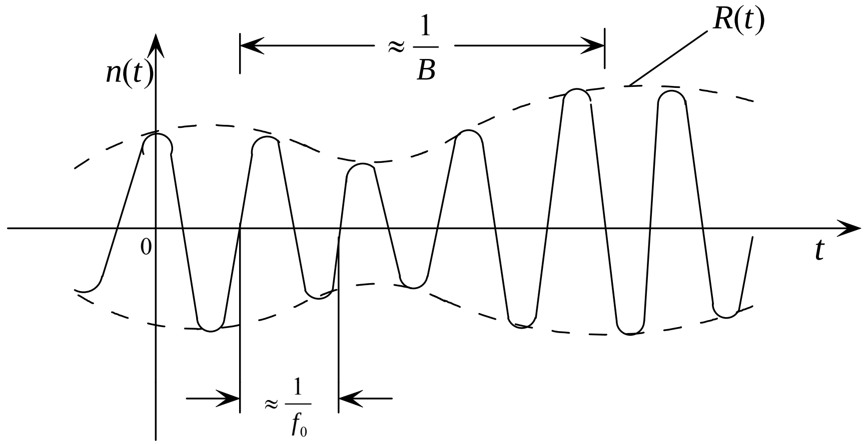
- Result 5：If n(t) is a Gaussian process, so are $n_c(t)$ and $n_s(t)$

Envelop and Phase

- Angular representation of n(t)

$$n(t) = R(t)\cos\left[\omega_0 t + \phi(t)\right]$$

where
$$\begin{cases} R(t) = \sqrt{n_c^{\ 2}(t) + n_s^{\ 2}(t)} & \text{envelop} \\ \phi(t) = \tan^{-1}\dfrac{n_s(t)}{n_c(t)} \quad \left[0 \le \phi(t) \le 2\pi\right] & \text{phase} \end{cases}$$



- Let n(t) be a zero-mean, stationary Gaussian process, find the statistics of the envelop and phase
- Result：
  - Envelop follows Rayleigh distribution while phase follows uniform distribution

$$\begin{cases} f(R)=\int_0^{2\pi} f(R,\phi)d\phi=\dfrac{R}{\sigma^2}\exp\left\{-\dfrac{R^2}{2\sigma^2}\right\} & R\geq 0 \\ f(\phi)=\int_0^{\infty} f(R,\phi)dR=\dfrac{1}{2\pi} & 0\leq\phi\leq 2\pi \end{cases}$$

Proof?

  - For the same t, the envelop variable R and phase variable ϕ are independent (but not the two processes)

Homework 1

- Textbook Chapter 2: **2.7(3)(4), 2.13(6)(13)(16)**
- Textbook Chapter 5: **5.5, 5.15, 5.22, 5.28, 5.44, 5.49**
- Due: **in class** on Sep. 28 (**next Wednesday**)