Principles of Communications

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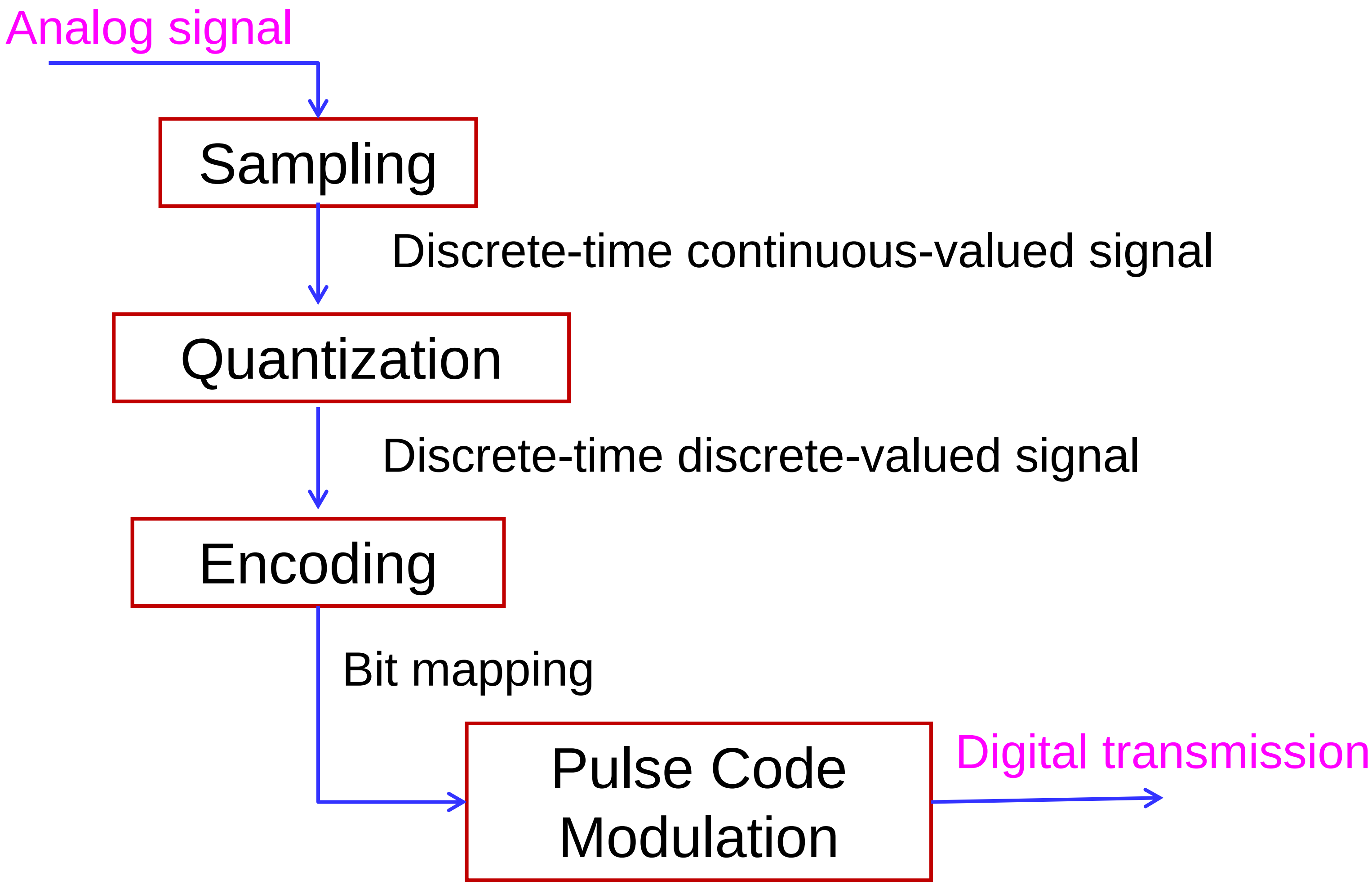
Chapter 4: Analog-to-Digital Conversion

Textbook: 7.1 – 7.4

Outline

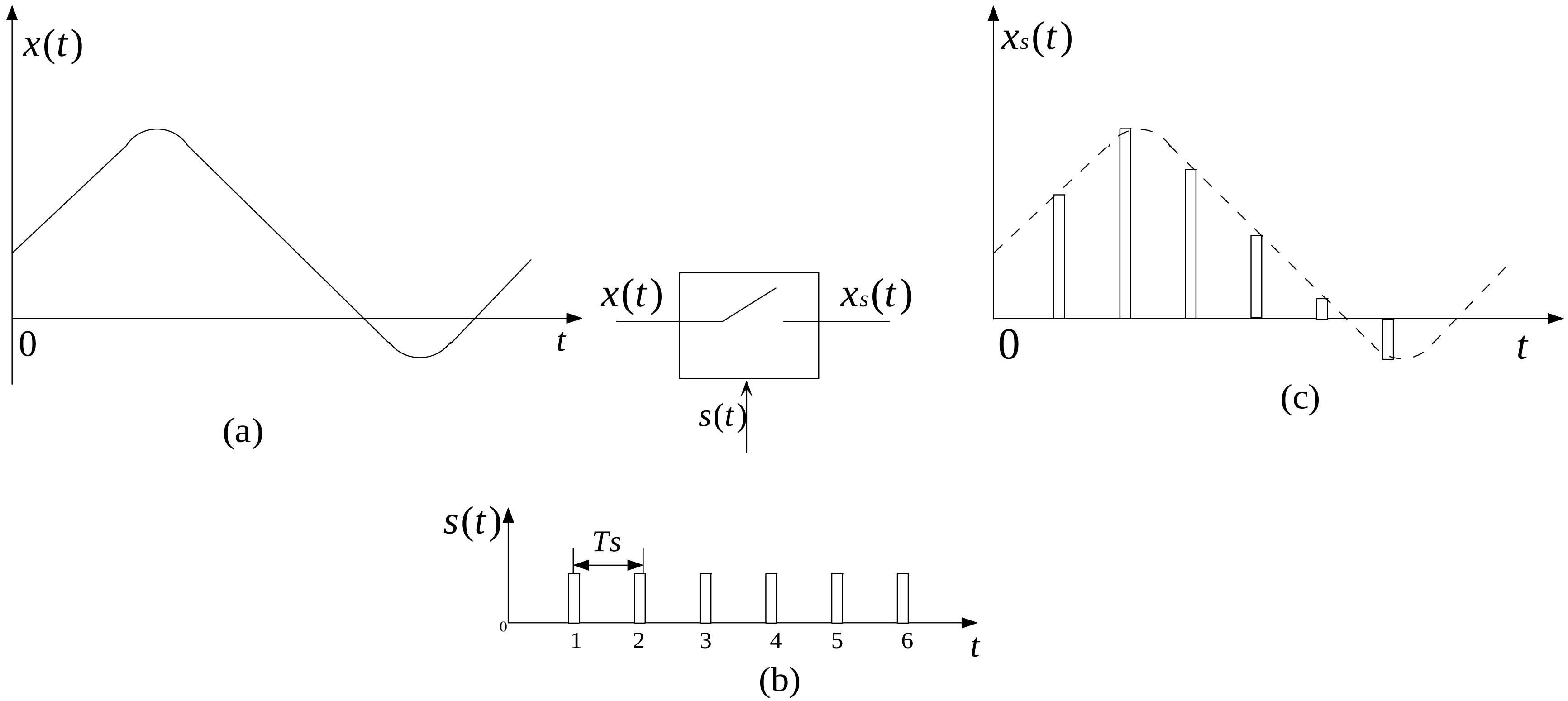

Analog signal
Sampling
Discrete-time continuous-valued signal
Quantization
Discrete-time discrete-valued signal
Encoding
Bit mapping
Pulse Code Modulation
Digital transmission

Sampling

Sampling Theorem:

- Let the signal $x(t)$ have a bandwidth W, i.e., let $X(f) \equiv 0, \text{ for } |f| \geq W$. Let $x(t)$ be sampled at time interval $T_s \leq \frac{1}{2W}$ to yield the sequence $\{x(nT_s)\}_{n=-\infty}^{\infty}$. Then it is possible to reconstruct the original signal $x(t)$ from the sampled values.

Sampling Process



- The sampling process can be regarded a modulation process with carrier given by periodic impulses. It’s also called pulse modulation

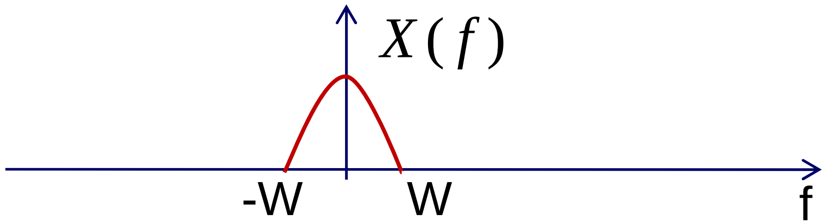
- The result of sampling can be written as

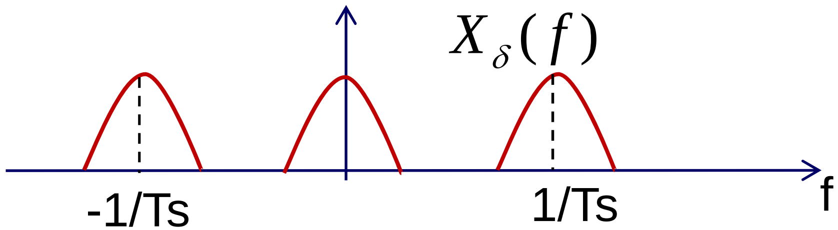
$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t-nT_s) = x(t)\sum_{n=-\infty}^{\infty}\delta(t-nT_s)$$

- Taking Fourier transform

$$X_\delta(f) = X(f) * \frac{1}{T_s}\sum_{n=-\infty}^{\infty}\delta\left(f-\frac{n}{T_s}\right) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} X\left(f-\frac{n}{T_s}\right)$$

- If $T_s > \frac{1}{2W}$ or $f_s = \frac{1}{T_s} < 2W$ then the replicated spectrum of x(t) overlaps and reconstruction is not possible, known as aliasing error





- The minimum sampling rate $f_s = 2W$ is known as Nyquist sampling rate

Reconstruction

- To get the original signal back, it is sufficient to filter the sampled signal through a LPF with frequency response

$$H(f) = \begin{cases} 1 & |f| < W \\ 0 & |f| \geq \frac{1}{T_s} - W \end{cases}$$

- The reconstruction is

$$x(t) = \sum_{n=-\infty}^{\infty} 2W' T_s x(nT_s) \operatorname{sinc}\left[2W'(t - nT_s)\right]$$

$$\text{for} \quad W \leq W' \leq \frac{1}{T_s} - W$$

Quantization

- Quantization is a rounding process, each sampled signal point is rounded to the “nearest” value from a finite set of possible quantization levels.
- Scalar quantization
  - Each sample is quantized individually
- Vector quantization
  - Blocks of samples are quantized at a time

Scalar Quantization

- The set of real numbers R is partitioned into N disjoint subsets, denoted as Rk, each called quantization region
- For each region Rk, a representation point, called quantization level xk is chosen
- If the sampled signal belongs to region Rk, then it is represented by xk , i.e.

$$\boxed{Q(x) = x_k, \text{ for all } x \in \mathcal{R}_k, k = 1, \ldots, N}$$

- Quantization error

$$e(x, Q(x)) = x - Q(x)$$

Performance Measure of Quantization

- Signal-to-quantization noise ratio (SQNR) is defined by

$$SQNR = \frac{E(X^2)}{E\left(\left[X - Q(x)\right]^2\right)}$$

For random variable X

$$SQNR = \frac{\lim\limits_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x^2(t)dt}{\lim\limits_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} \left(x(t) - Q(x(t))\right)^2 dt}$$

For signal x(t)

Example

- The source X(t) is stationary Gaussian source with mean zero and power spectral density

$$S_x(f) = \begin{cases} 2 & |f| < 100Hz \\ 0 & \text{otherwise} \end{cases}$$

- It is sampled at the Nyquist rate and each sample is quantized using the 8-level quantizer with

$$a_0 = -\infty, a_1 = -60, a_2 = -40, a_3 = -20, a_4 = 0, a_5 = 20, a_6 = 40, a_7 = 60$$

$$x_1 = -70, x_2 = -50, x_3 = -30, x_4 = -10, x_5 = 10, x_6 = 30, x_7 = 50, x_8 = 70$$

- What is the resulting distortion and rate?
- What is the SQNR?

Uniform Quantizer

- Let the range of the input samples is [-a, a] and the number of quantization levels is N = 2^v. Then the length of each quantization region is given by

$$\Delta = \frac{2a}{N} = \frac{a}{2^{v-1}}$$

- Quantized values are the midpoints of quantization regions.
- Assuming that the quantization error is uniformly distributed on $(-\frac{\Delta}{2}, \frac{\Delta}{2})$. Then

$$E[e^2] = \int_{-\Delta/2}^{\Delta/2} \frac{1}{\Delta} x^2 dx = \frac{\Delta^2}{12} = \frac{a^2}{2N^2} = \frac{a^2}{3 \cdot 4^v}$$

$$\boxed{SQNR = \frac{P_X}{E[e^2]} = \frac{3 \cdot 4^v P_X}{a^2} \stackrel{dB}{=} 10 \log_{10} \frac{P_X}{a^2} + 6v + 4.8}$$

One extra bit increases the SQNR by 6 dB!

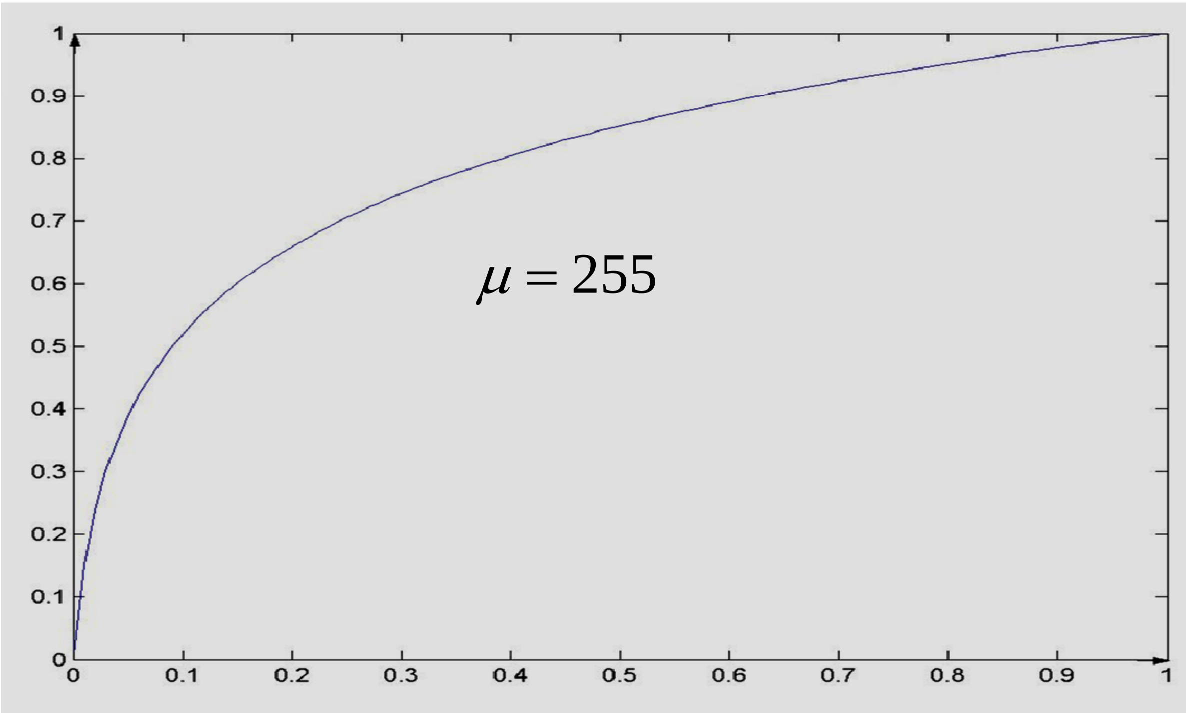
Nonuniform Quantizer

- If we relax the condition that the quantization regions be of equal length, then we can minimize the distortion with less constraints; therefore, the resulting quantizer will perform better than a uniform quantizer
- The usual method of nonuniform quantization is to first pass the samples through a nonlinear filter and then perform a uniform quantization => Companding
- For speech coding, higher probability for smaller amplitude and lower probability for larger amplitude
  - μ law compander
  - A-law compander

Compander

- μ law compander

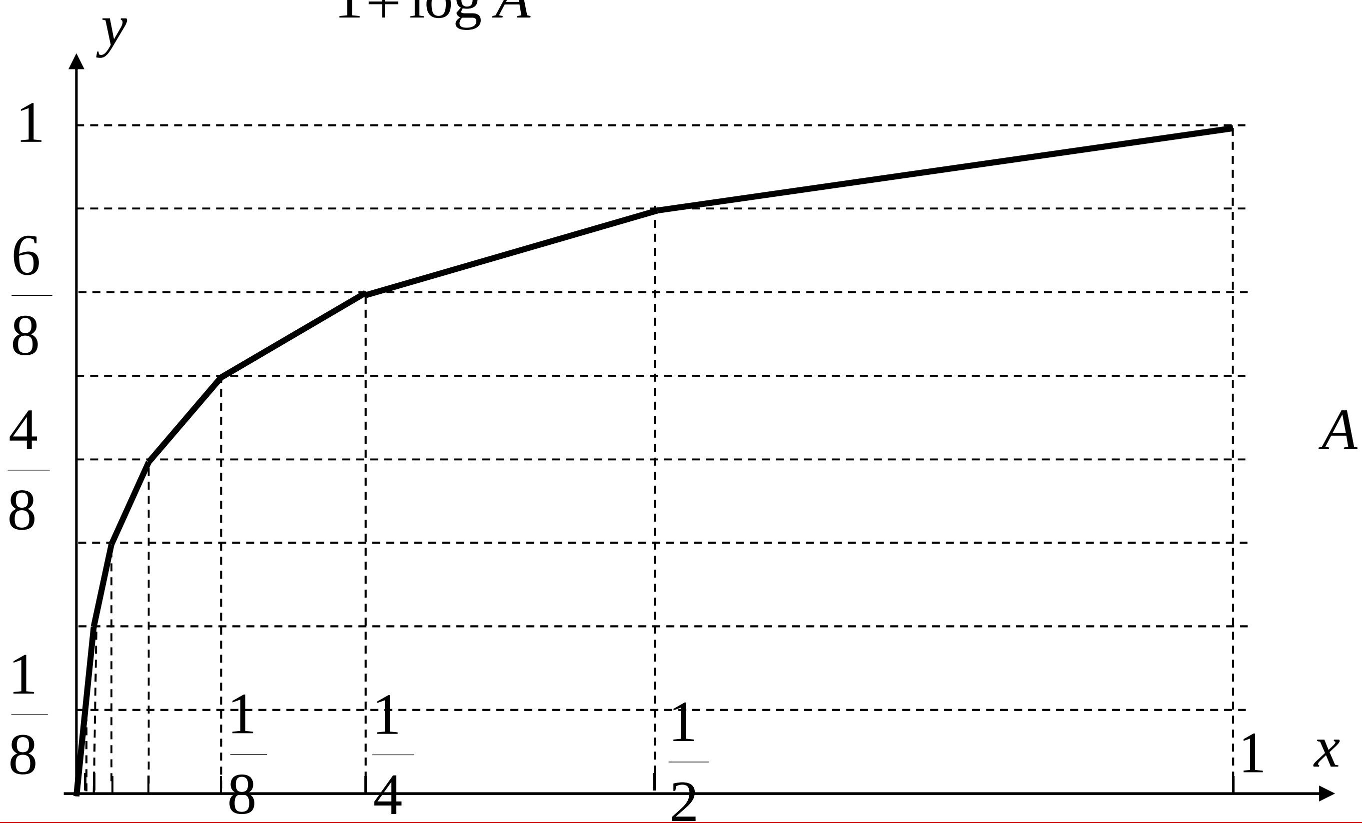
$$g(x) = \frac{\log\left(1+\mu|x|\right)}{\log(1+\mu)}\operatorname{sgn}(x), \quad |x| \le 1$$



Compander

- A law compander

$$g(x) = \frac{1 + \log A|x|}{1 + \log A} \operatorname{sgn}(x), \quad |x| \le 1$$



$A = 87.6$

Optimal Quantizer

- Lloyd-Max Conditions
  - The boundaries of the quantization regions are the midpoints of the corresponding quantized values
  - The quantized values are the centroids of the quantization regions.

Vector Quantization

- The idea of vector quantization is to take blocks of source outputs of length n, and design the quantizer in the n-dim Euclidean space, rather than doing the quantization based on single samples in a one-dim space
- Optimal vector quantizer to minimize distortion
  - Region Ri is the set of all points in the n-dim space that are closer to xi than any other xj, for all j\= I; i.e.

$$R_i = \left\{ \mathbf{x} \in R^n : \left\| \mathbf{x} - \mathbf{x}_i \right\| < \left\| \mathbf{x} - \mathbf{x}_j \right\|, \forall j \neq i \right\}$$

  - xi is the centroid of the region Ri, i.e.

$$\mathbf{x}_i = \frac{1}{P(\mathbf{x} \in R_i)} \int_{R_i} \mathbf{x} f_{\mathbf{x}}(\mathbf{x}) d\mathbf{x}$$

4.3 Encoding

- The encoding process is to assign v bits to N=2^v quantization levels.
- Since there are v bits for each sample and fs samples/second, we have a bit rate of

$$R = v f_s \text{ bits/second}$$

- Natural binary coding
  - Assign the values of 0 to N-1 to different quantization levels in order of increasing level value.
- Gray coding
  - Adjacent levels differ only in one bit

Examples

- Natural binary code (NBC), folded binary code (FBC), 2-complement code (2-C), 1-complement code (1-C), and Gray code

Level no	NBC	FBC	2-C	Gray code	Amplitude level
7	111	011	011	100	3.5
6	110	010	010	101	2.5
5	101	001	001	111	1.5
4	100	000	000	110	0.5
3	011	100	100	010	-0.5
2	010	101	111	011	-1.5
1	001	110	110	001	-2.5
0	000	111	101	000	-3.5

Pulse Code Modulation (PCM) Systems

- Block diagram of a PCM system

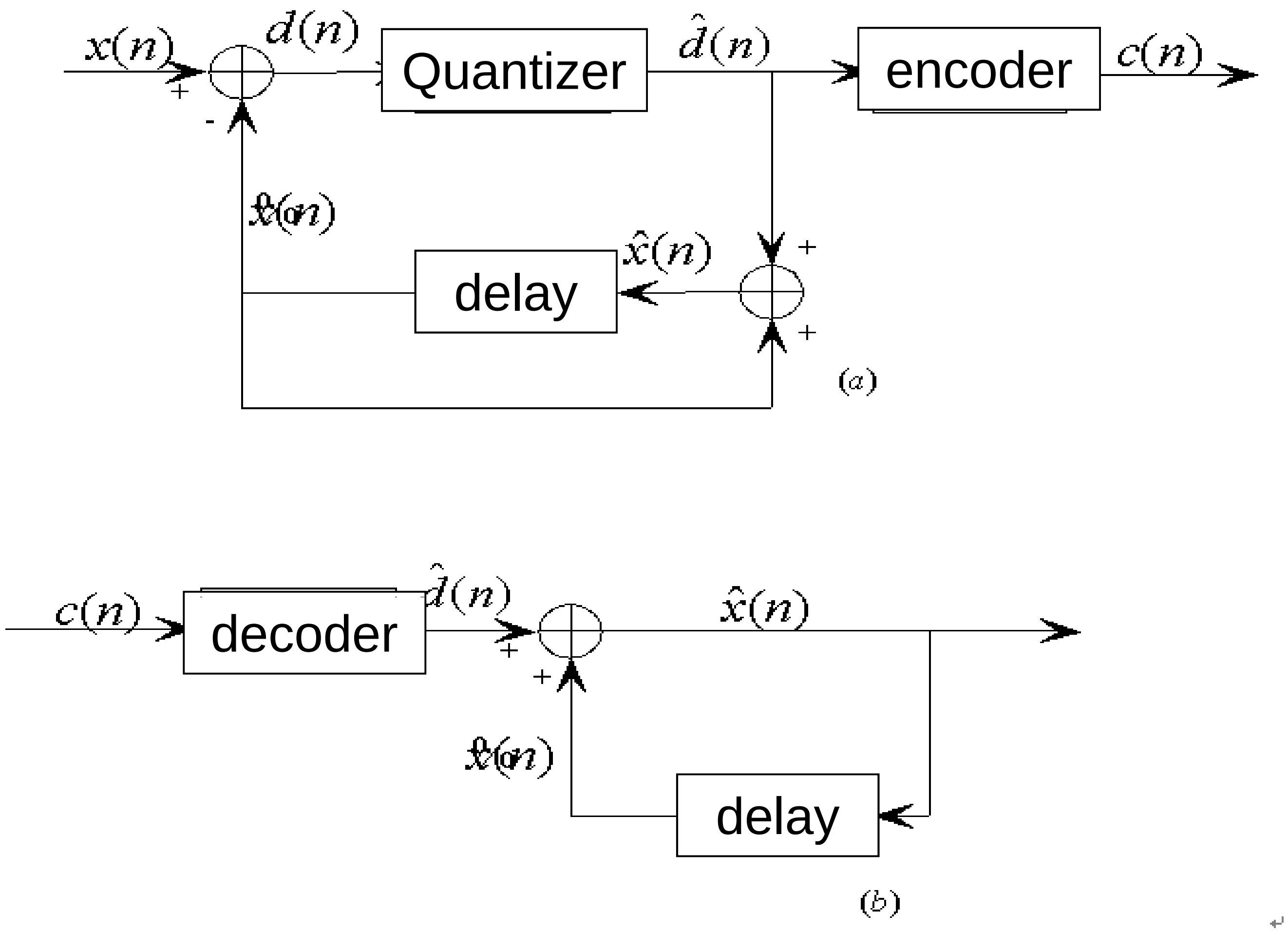
- Bandwidth requirement:
  - If a signal has a bandwidth of W and v bits are used for each sampled signal, then

$$BW_{req} = \frac{R}{2} = vW \text{ Hz}$$

Differential PCM (DPCM)

- For a bandlimited random process, the sampled values are usually correlated random variables
- This correlation can be employed to improve the performance
- Differential PCM: quantize the difference between two adjacent samples.
- As the difference has small variation, to achieve a certain level of performance, fewer bits are required

DPCM


$x(n)$
$d(n)$
Quantizer
$\hat{d}(n)$
encoder
$c(n)$
+
-
$\tilde{x}(n)$
delay
$\hat{x}(n)$
+
+
(a)
$c(n)$
decoder
$\hat{d}(n)$
+
+
$\hat{x}(n)$
$\tilde{x}(n)$
delay
(b)

Delta Modulation (DM)

- DM is a simplified version of DPCM, where the quantizer is a two-level quantizer with magnitude $\pm\Delta$
- In DM, only 1-bit per symbol is employed. So adjacent samples must have high correlation.
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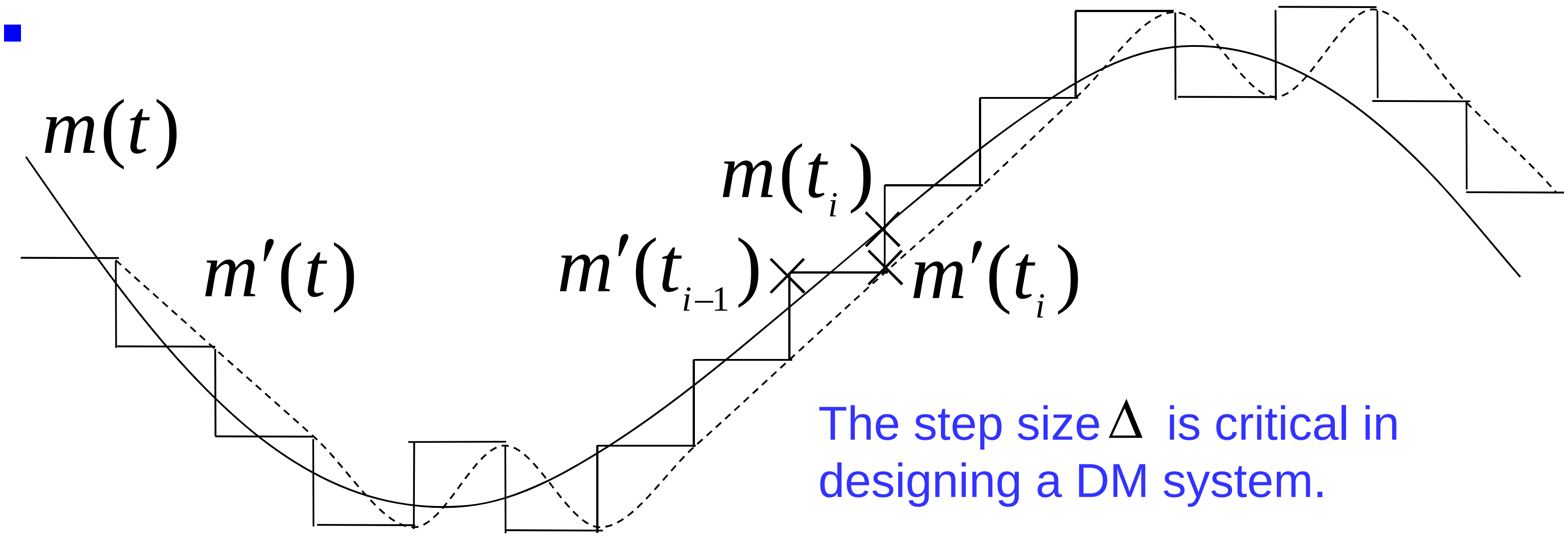

The step size Δ is critical in designing a DM system.