Principles of Communications

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Chapter 6: Signal Space Representation

:Textbook: Chapter 8.1

Signal Space Concepts

- The key to analyzing and understanding the performance of digital transmission is the realization that
  - **signals used in communications can be expressed and visualized graphically**
- Thus, we need to understand **signal space concepts** as applied to digital communications

Traditional Bandpass Signal Representations

- Baseband signals are the message signal generated at the source
- Passband signals (also called bandpass signals) refer to the signals after modulating with a carrier. The bandwidth of these signals are usually small compared to the carrier frequency f_c
- Passband signals can be represented in three forms

  - ✓ Magnitude and Phase representation
  - ✓ Quadrature representation
  - Complex Envelop representation

Magnitude and Phase Representation

$$s(t) = a(t)\cos\left[2\pi f_c t + \theta(t)\right]$$

Where $a(t)$ is the magnitude of $s(t)$
and $\theta(t)$ is the phase of s(t)

Quadrature or I/Q Representation

$$s(t) = x(t)\cos(2\pi f_c t) - y(t)\sin(2\pi f_c t)$$

where x(t) and y(t) are real-valued baseband signals called the in-phase and quadrature components of s(t).

Signal space is a more convenient way than I/Q representation to study modulation scheme

Vectors and Space

- Consider an n-dimensional space with unity basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ (think of the x-y-z axis in a coordinate system)
- Any vector $\mathbf{a}$ in the space can be written as

$$\mathbf{a} = \sum_{i=1}^{n} a_i \mathbf{e}_i \quad \Rightarrow \quad \mathbf{a} = (a_1, a_2, \ldots, a_n)$$

$n \triangleq$ Dimension = Minimum number of vectors that is necessary and sufficient for representation of any vector in space

- **Definitions**:
  - Inner Product $\langle \mathbf{a}, \mathbf{b} \rangle = \mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{n} a_i b_i$
  - **a** and **b** are Orthogonal if $\mathbf{a} \cdot \mathbf{b} = 0$
  - $\|\mathbf{a}\| = \sqrt{\langle \mathbf{a}, \mathbf{a} \rangle} = \sqrt{\sum_{i=1}^{n} a_i^2}$ = Norm of **a**
  - A set of vectors are orthonormal if they are mutually orthogonal and all have unity norm

Basis Vectors

- The set of basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ of a space are chosen such that:
  - Should be complete or span the vector space: any vector $\mathbf{a}$ can be expressed as a linear combination of these vectors.
  - Each basis vector should be orthogonal to all others

$$\mathrm{e}_i \cdot \mathrm{e}_j = 0, \ \forall i \neq j$$

  - Each basis vector should be normalized: $\|e_i\| = 1, \ \forall i$
- A set of basis vectors satisfying these properties is also said to be a complete orthonormal basis
- In an **n-dim** space, we can have at most n basis vectors

Signal Space

- Basic Idea: If a signal can be represented by n-tuple, then it can be treated in much the same way as a n-dim vector.
- Let $\phi_1(t)$, $\phi_2(t)$,…., $\phi_n(t)$ be *n* signals
- Consider a signal x(t) and suppose that

$$x(t) = \sum_{i=1}^{n} \phi_i(t)$$

- If every signal can be written as above $\Rightarrow$ $\{\phi_1(t), \ldots, \phi_n(t)\}$ ~ **basis functions** and we have **a n-dim signal space**

Orthonormal Basis

- Signal set $\{\phi_k(t)\}^n$ is an **orthogonal** set if

$$\int_{-\infty}^{\infty} \phi_j(t)\phi_k(t)dt = \begin{cases} 0 & j \neq k \\ c_j & j = k \end{cases}$$

- If $c_j \equiv 1 \ \forall_j \Rightarrow \{\phi_k(t)\}$ is an **orthonormal** set.
- In this case,

$$x_k = \int_{-\infty}^{\infty} x(t)\phi_k(t)dt$$

$$x(t) = \sum_{i=1}^{n} x_i \phi_i(t)$$

$$\mathbf{x} = (x_1, x_2, \ldots, x_n)$$

Key Property

- Given the set of the orthonormal basis

$$\{\phi_1(t), \ldots, \phi_n(t)\}$$

- Let $x(t)$ and $y(t)$ be represented as

$$x(t) = \sum_{i=1}^{n} x_i \phi_i(t) \;, \qquad y(t) = \sum_{i=1}^{n} y_i \phi_i(t)$$

with $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $\mathbf{y} = (y_1, y_2, \ldots, y_n)$

➢ Then the inner product of $\mathbf{x}$ and $\mathbf{y}$ is

$$\boxed{\mathbf{x} \cdot \mathbf{y} = \int_{-\infty}^{\infty} x(t) y(t) dt}$$

- Proof

$$\int_{-\infty}^{\infty} x(t)y(t)dt = \int_{-\infty}^{\infty} \left[\sum_{i=1}^{n} x_i\phi_i(t)\right] \left[\sum_{j=1}^{n} y_j\phi_j(t)\right] dt$$

$$= \sum_{k=1}^{n} x_k y_k \triangleq \mathbf{x} \cdot \mathbf{y}$$

Since $$\int_{-\infty}^{\infty} \phi_i(t)\phi_j(t)dt = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

➢ E_x = Energy of x(t) = $\int_{-\infty}^{\infty} x^2(t)dt$

⇒ $E_x = \mathbf{x} \cdot \mathbf{x} = \|\mathbf{x}\|^2$

Basis Functions for a Signal Set

- Consider a set of M signals (M-ary symbol) $\{s_i(t), i=1,2,\ldots,M\}$ with finite energy. That is

$$\int_{-\infty}^{\infty} s_i^2(t)dt < \infty$$

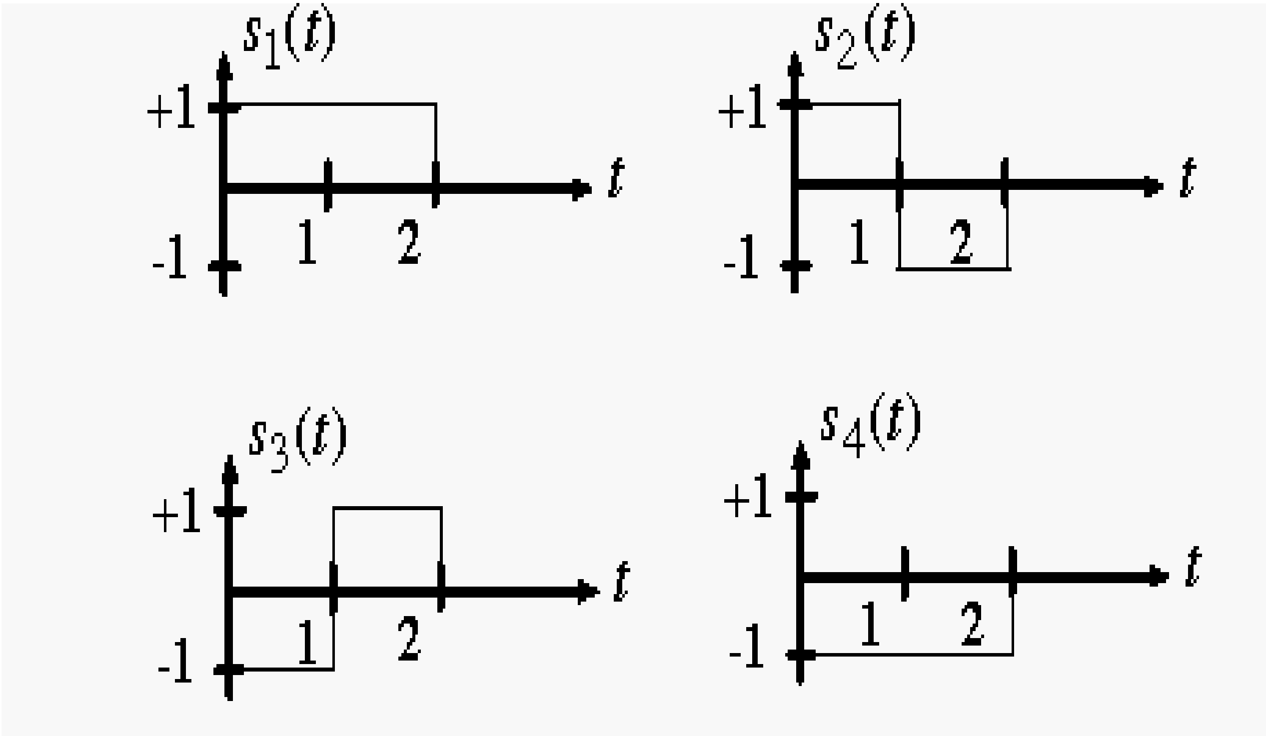
- Then, we can express each of these waveforms as weighted linear combination of orthonormal signals

$$s_i(t) = \sum_{j=1}^{N} s_{ij}\phi_j(t) \qquad \text{for } i = 1, \ldots, M$$

where $N \leq M$ is the dimension of the signal space and $\{\phi_j(t)\}_1^N$ are called the orthonormal basis functions

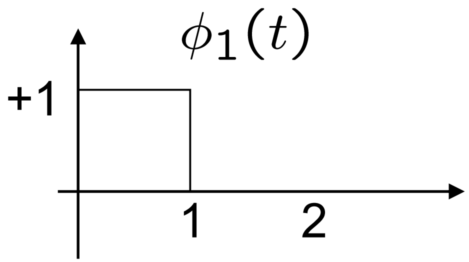
Example 1

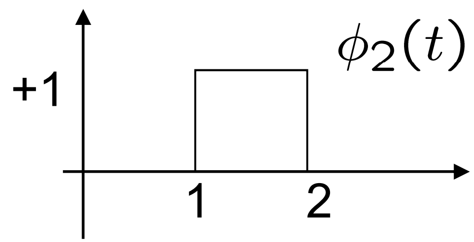
Consider the following signal set:



Example 1 (Cont'd)

- By inspection, the signals can be expressed in terms of the following two basis functions:





$$s_1(t) = 1 \cdot \phi_1(t) + 1 \cdot \phi_2(t) \qquad s_3(t) = -1 \cdot \phi_1(t) + 1 \cdot \phi_2(t)$$

$$s_2(t) = 1 \cdot \phi_1(t) - 1 \cdot \phi_2(t) \qquad s_4(t) = -1 \cdot \phi_1(t) - 1 \cdot \phi_2(t)$$

- Not that the basis is orthogonal $\int_{-\infty}^{\infty} \phi_1(t)\phi_2(t)dt = 0$
- Also note that each these functions have unit energy

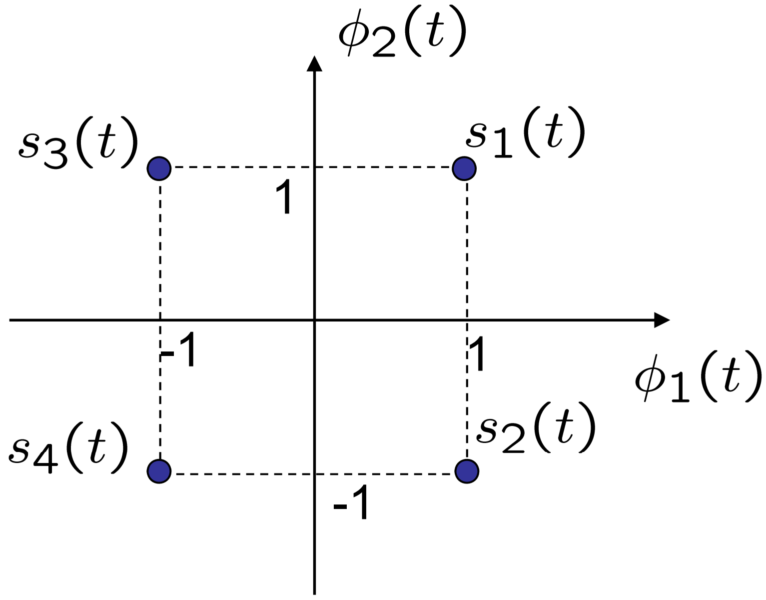
$$\int_{-\infty}^{\infty} |\phi_1(t)|^2 dt = \int_{-\infty}^{\infty} |\phi_2(t)|^2 dt = 1$$

- We say that they form an orthonormal basis

Example 1 (Cont'd)

Constellation diagram:

- A representation of a digital modulation scheme in the signal space
- Axes are labeled with $\phi_1(t)$ and $\phi_2(t)$
- Possible signals are plotted as points, called constellation points



Example 2

- Suppose our signal set can be represented in the following form

$$s(t) = \pm\sqrt{\frac{2}{T}}\cos(2\pi f_c t) \pm \sqrt{\frac{2}{T}}\sin(2\pi f_c t)$$

with $t \in [0, T)$ and $f_c T >> 1$

- We can choose the basis functions as follows

$$\phi_1(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_c t) \qquad \phi_2(t) = \sqrt{\frac{2}{T}}\sin(2\pi f_c t)$$

$t \in [0, T)$

Example 2 (Cont'd)

- Since

$$\int_0^T \phi_1(t)\phi_2(t)dt = \int_0^T \sqrt{\frac{2}{T}}\cos(2\pi f_c t)\cdot\sqrt{\frac{2}{T}}\sin(2\pi f_c t)dt$$

$$= \frac{2}{T}\int_0^T \frac{1}{2}\left[\sin(0) + \sin(4\pi f_c t)\right]dt$$

$$= \frac{-1}{4\pi f_c T}\left[\cos(4\pi f_c t)\right]_0^T \approx 0, \text{ for } f_c T >> 1$$

and

$$\int_0^T |\phi_1(t)|^2 dt = \int_0^T |\phi_2(t)|^2 dt = \frac{2}{T}\int_0^T \frac{1}{2}[1 + \cos(4\pi f_c t)]dt \approx 1$$

- The basis functions are thus orthogonal and they are also normalized

Example 2 (Cont'd)

- These basis functions are quite common and can describe various modulation schemes
- Example 2 is QPSK modulation. Its constellation diagram is identical to Example 1

Notes on Signal Space

- Two entirely different signal sets can have the same geometric representation.
- The underlying geometry will determine the performance and the receiver structure for a signal set
- In previous examples, we were able to guess the correct basic functions
- In general, is there any method which allows us to find a complete orthonormal basis for an arbitrary singal set?
  - Gram-Schmidt Orthogonalization (GSO) Procedure

Gram Schmidt Orthogonalization (GSO) Procedure

- Suppose we are given a signal set

$$\{s_1(t), \ldots, s_M(t)\}$$

- Find the orthogonal basis functions for this signal set $\{\phi_1(t), \ldots, \phi_K(t)\}$

where $K \leq M$

Step 1: Construct the First Basis Function

- Compute the energy in signal 1:

$$E_1 = \int_{-\infty}^{\infty} s_1^2(t)dt$$

- The first basis function is just a normalized version of $s_1(t)$

$$\phi_1(t) = \frac{1}{\sqrt{E_1}} s_1(t)$$

$$\Rightarrow \quad s_1(t) = s_{11}\phi_1(t) = \sqrt{E_1}\phi_1(t)$$

$$s_{11} = \int_{-\infty}^{\infty} s_1(t)\phi_1(t)dt = \sqrt{E_1}$$

Step 2: Construct the Second Basis Function

- Compute correlation between signal 2 and basic function 1

$$s_{21} = \int_{-\infty}^{\infty} s_2(t)\phi_1(t)dt$$

- Subtract off the correlation portion

$$g_2(t) = s_2(t) - s_{21}\phi_1(t)$$ ⇒ $g_2(t)$ is orthogonal to $\phi_1(t)$

- Compute the energy in the remaining portion

$$E_{g_2} = \int_{-\infty}^{\infty} [g_2(t)]^2 \, dt$$

- Normalize the remaining portion

$$\phi_2(t) = \frac{1}{\sqrt{E_{g_2}}} g_2(t)$$

⇒ $$s_{22} = \int_{-\infty}^{\infty} s_2(t)\phi_2(t)dt = \sqrt{E_{g_2}}$$

Step 3: Construct Successive Basis Functions

- For signal $s_k(t)$, compute $s_{ki} = \int_{-\infty}^{\infty} s_k(t)\phi_i(t)dt$
- Define $g_k(t) = s_k(t) - \sum_{i=1}^{k-1} s_{ki}\phi_i(t)$
- Energy of $g_k(t)$: $E_{g_k} = \int_{-\infty}^{\infty} [g_k(t)]^2 \, dt$
- *k*-th basis function: $\phi_k(t) = \frac{1}{\sqrt{E_{g_k}}} g_k(t)$
- In general

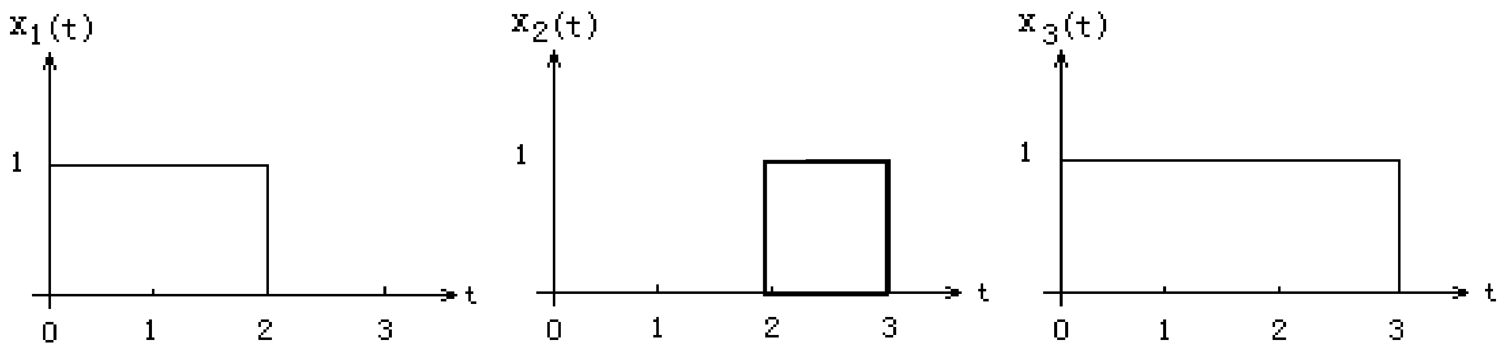
$$s_{kk} = \int_{-\infty}^{\infty} s_k(t)\phi_k(t)dt = \sqrt{E_{g_k}}$$

Summary of GSO Procedure

- 1st basis function is normalized version of the first signal
- Successive basis functions are found by removing portions of signals that are correlated to previous basis functions and normalizing the result
- This procedure is repeated until all basis functions are found
- If $g_k(t) = 0$, no new basis functions is added
- The order in which signals are considered is arbitrary

Example: GSO

1) Use the Gram-Schmidt procedure to find a set orthonormal basis functions corresponding to the signals show below



2) Express x_1, x_2, and x_3 in terms of the orthonormal basis functions found in part 1)

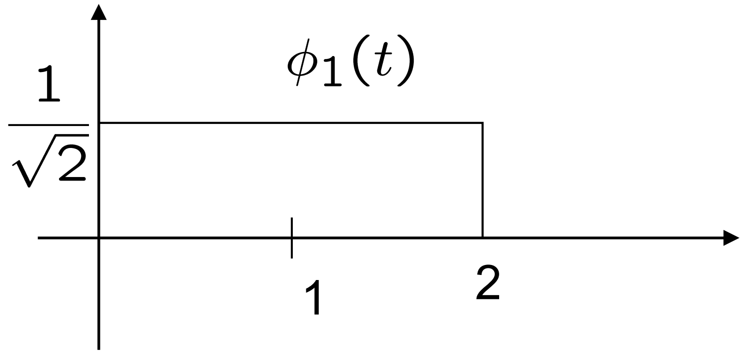
3) Draw the constellation diagram for this signal set

Solution: 1)

Step 1: $E_1 = \int_{-\infty}^{\infty} x_1^2(t)dt = 2$

$$\phi_1(t) = \frac{1}{\sqrt{2}} x_1(t)$$

$$x_{11} = \sqrt{2}$$



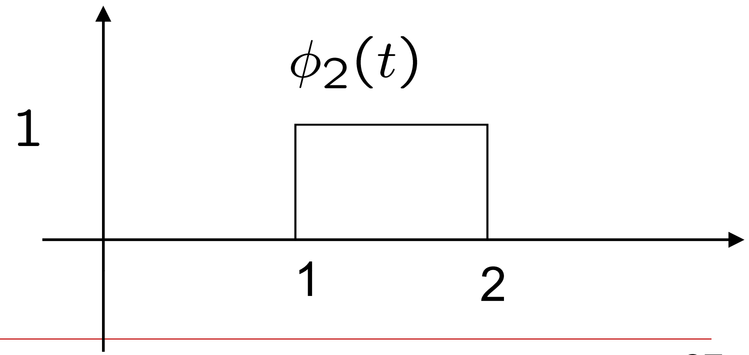
Step 2:

$$x_{21} = \int_{-\infty}^{\infty} x_2(t)\phi_1(t)dt = 0$$

$g_2(t) = x_2(t)$ and $E_{g_2} = 1$

$$\phi_2(t) = x_2(t)$$

$$x_{22} = 1$$



Solution: 1) (Cont'd)

- Step 3: $$x_{31} = \int_{-\infty}^{\infty} x_3(t)\phi_1(t)dt = \sqrt{2}$$

$$x_{32} = \int_{-\infty}^{\infty} x_3(t)\phi_2(t)dt = 1$$

$$g_3(t) = x_3(t) - x_{31}f_1(t) - x_{32}f_2(t) = 0$$

=> No more new basis functions

Procedure completes

$$\begin{cases} \phi_1(t) = \dfrac{1}{\sqrt{2}}x_1(t) \\ \phi_2(t) = x_2(t) \end{cases}$$

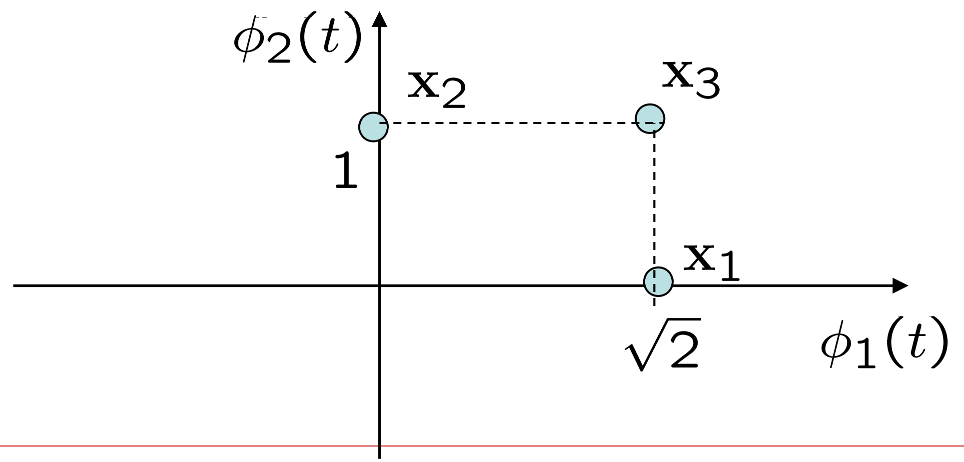
Solution: 2) and 3)

- Express x_1, x_2, x_3 in basis functions

$$x_1(t) = \sqrt{2}\phi_1(t), \quad x_2(t) = \phi_2(t)$$

$$x_3(t) = \sqrt{2}\phi_1(t) + \phi_2(t)$$

- Constellation diagram



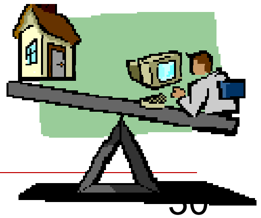
Exercise

Given a set of signals (8PSK modulation)

$$s_i(t) = A\cos\left(2\pi f_c t + \frac{\pi}{4}i\right)$$

$$i = 0, 1, \ldots, 7 \quad \text{and } 0 \le t < T$$

- Find the orthonormal basis functions using Gram Schmidt procedure
- What is the dimension of the resulting signal space ?
- Draw the constellation diagram of this signal set



Notes on GSO Procedure

- A signal set may have many different sets of basis functions
- A change of basis functions is essentially a rotation of the signal points around the origin.
- The order in which signal are used in the GSO procedure affect the resulting basis functions
- The choice of basis functions does not affect the performance of the modulation scheme