Principles of Communications

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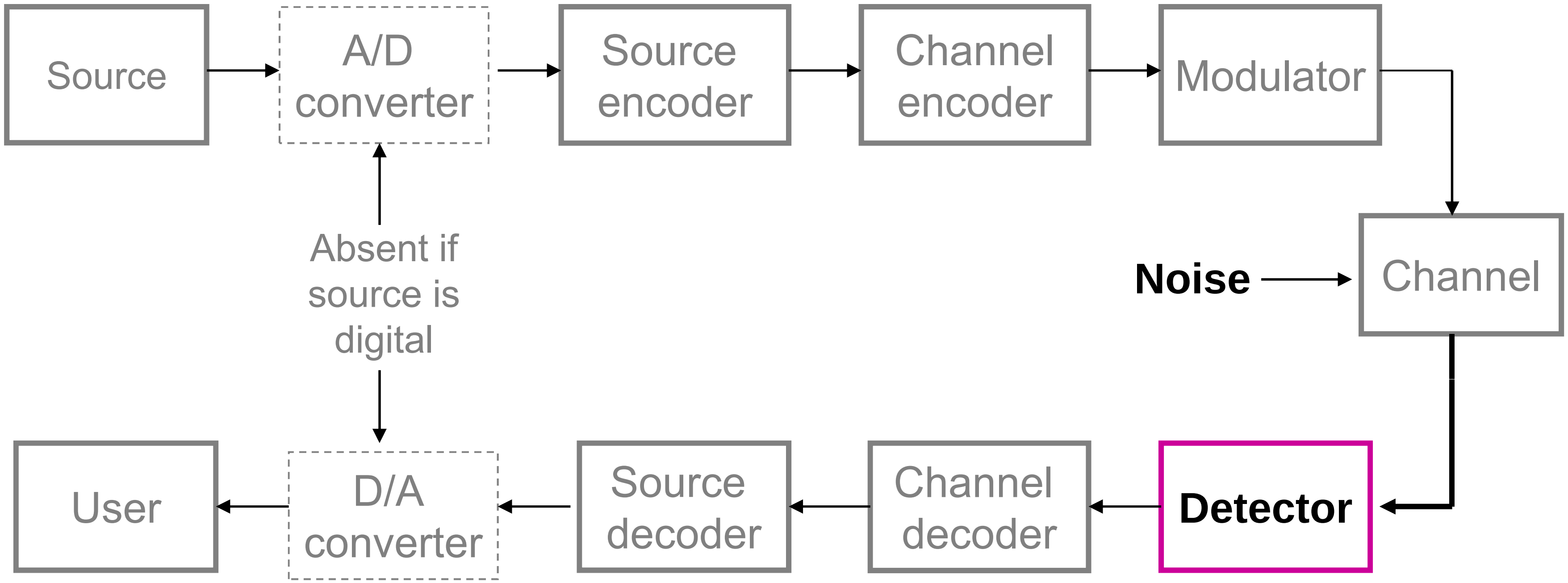
Chapter 7: Optimal Receivers

Textbook: Chapter 8.1-8.3

Topics to be Covered



- Detection theory
- Optimal receiver structure
- Matched filter
- Decision regions
- Error probability analysis

6.1 Statistical Decision Theory

- Demodulation and decoding of signals in digital communications is directly related to Statistical decision theory
- In the general setting, we are given a finite set of possible hypotheses about an experiment, along with observations related statistically to the various hypotheses.
- The theory provides rules for making the best possible decision (according to some performance criterion) about which hypothesis is likely to be true
- In digital communications, hypotheses are the possible messages and observations are the output of a channel
- A decision on the transmitted data is made based on the observed values of the channel output
- We are interested in the best decision making rule in the sense of minimizing the probability of error

Detection Theory

- Given M possible hypotheses H_i (signal m_i) with probability

$$P_i = P(m_i) \quad , \quad i = 1, 2, \ldots, M$$

  - P_i represents the prior knowledge concerning the probability of the signal m_i – Prior Probability

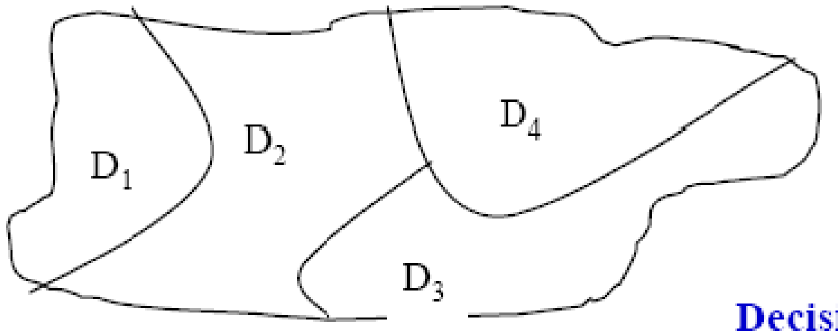
- The observation is some collection of N real values, denoted by $\vec{r} = (r_1, r_2, \ldots, r_N)$ with conditional pdf

  $f(\vec{r}|m_i)$ -- conditional pdf of observation $\vec{r}$ given the signal m_i

- Goal: Find the best decision-making algorithm in the sense of minimizing the probability of decision error.

Observation Space

- In general, $\vec{r}$ can be regarded as a point in some observation space
- Each hypothesis H_i is associated with a decision region D_i:
- The decision will be in favor of H_i if $\vec{r}$ is in D_i
- Error occurs when a decision is made in favor of another when the signals falls outside the decision region D_i



Decision Space (M=4)

MAP Decision Criterion

- Consider a decision rule based on the computation of the posterior probabilities defined as

  $P(m_i|\vec{r})$ = P(signal m_i was transmitted given $\vec{r}$ observed)

  for $i = 1, \ldots, M$

  - Known as a posterior since the decision is made after (or given) the observation
  - Different from the a prior where some information about the decision is known in advance of the observation
- By Bayes' Rule

$$P(m_i|\vec{r}) = \frac{P_i f(\vec{r}|m_i)}{f(\vec{r})}$$

MAP Decision Criterion (cont'd)

- Since our criterion is to minimize the probability of detection error given $\vec{r}$, we deduce that the optimum decision rule is to choose $\hat{m} = m_k$ if and only if $P(m_i|\vec{r})$ is maximum for $i = k$
- Equivalently,

$$\boxed{\begin{array}{l}\text{Choose } \hat{m} = m_k \text{ if and only if} \\ P_k f(\vec{r}|m_k) \geq P_i f(\vec{r}|m_i); \text{ for all } i \neq k\end{array}}$$

- This decision rule is known as maximum a posterior or MAP decision criterion

ML Decision Criterion

- If $p_1=p_2= \ldots =p_M$, i.e. the signals $\{m_k\}$ are equiprobable, finding the signal that maximizes $P(m_k|\vec{r})$ is equivalent to finding the signal that maximizes $f(\vec{r}|m_k)$
- The conditional pdf $f(\vec{r}|m_k)$ is usually called the likelihood function. The decision criterion based on the maximum of $f(\vec{r}|m_k)$ is called the Maximum-Likelihood (ML) criterion.
- ML decision rule:

$$\boxed{\begin{array}{l}\text{Choose } \hat{m} = m_k \text{ if and only if} \\ \qquad f(\vec{r}|m_k) \geq f(\vec{r}|m_i);\ \text{for all } i \neq k\end{array}}$$

- In any digital communication systems, the decision task ultimately reverts to one of these rules

6.2 Optimal Receiver in AWGN Channel

- Transmitter transmits a sequence of symbols or messages from a set of M symbols m_1, m_2, …, m_M.

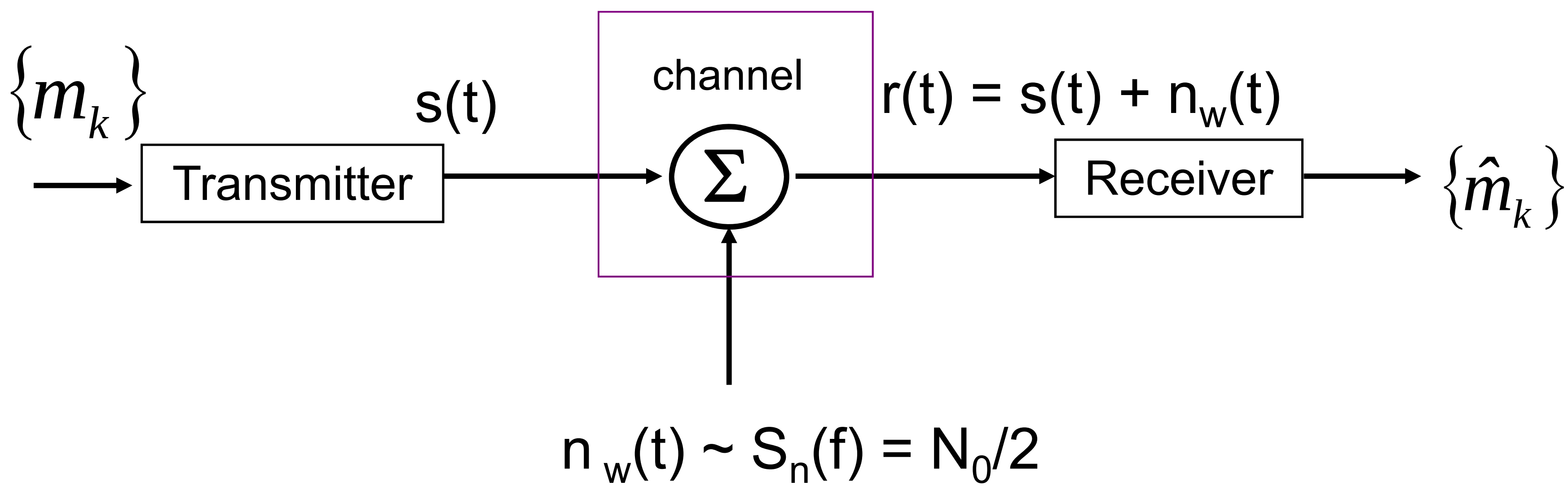
- The symbols are represented by finite energy waveforms $s_1(t)$, $s_2(t)$, …, $s_M(t)$, defined in the interval [0, T]

- Assume the symbols are transmitted with probability

$$p_1 = P(m_1),\ p_2 = P(m_2),\ p_M = P(m_M)$$

AWGN Channel Model

- The channel is assumed to corrupt the signal by additive white Gaussian noise (AWGN)
- Consider the following communication model



Signal Space Representation

- The signal space of $\{s_1(t), s_2(t), \ldots, s_M(t)\}$ is assumed to be of dimension N ($N \le M$)
- $\phi_k(t)$ for k = 1, …, N will denote an orthonormal basis function
- Then each transmitted signal waveform can be represented as

$$s_m(t) = \sum_{k=1}^{N} s_{mk}\phi_k(t) \quad \text{where} \quad s_{mk} = \int_0^T s_m(t)\phi_k(t)dt$$

- Note that the noise $n_w(t)$ can be written as

$$n_w(t) = \underbrace{n_0(t)}_{} + \underbrace{\sum_{k=1}^{N} n_k \phi_k(t)}_{\text{Projection of } n_w(t) \text{ on the N-dim space}} \quad \text{where } n_k = \int_0^T n_w(t)\phi_k(t)dt$$

Projection of $n_w(t)$ on the N-dim space

orthogonal to the space, falls outside the signal space spanned by $\{\phi_k(t), k = 1, \ldots N\}$

- The received signal can thus be represented as

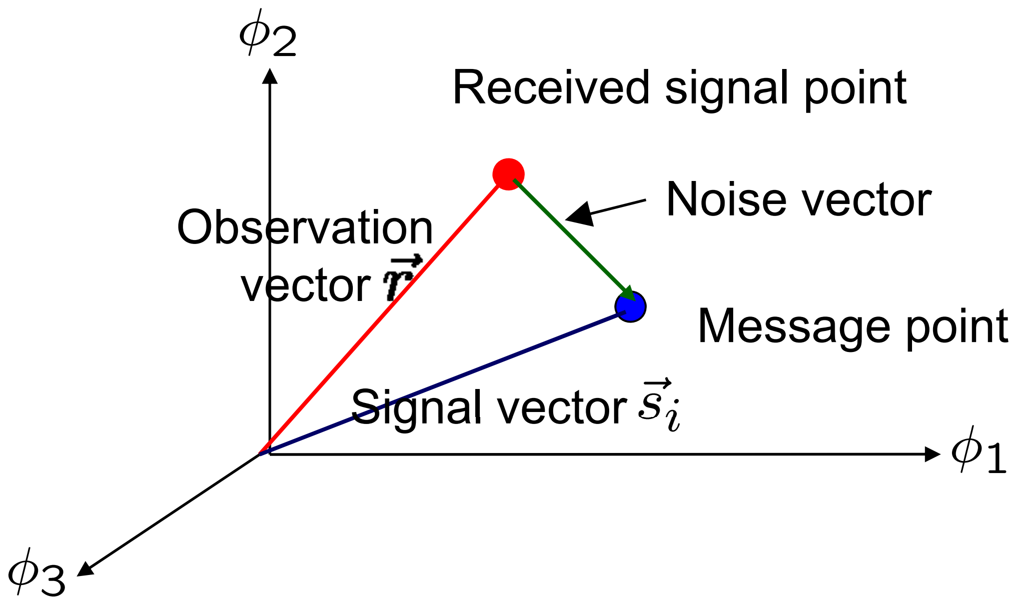
$$\begin{aligned} r(t) &= s(t) + n_w(t) \\ &= \sum_{k=1}^{N} s_{mk}\phi_k(t) + \sum_{k=1}^{N} n_k\phi_k(t) + n_0(t) \\ &= \underbrace{\sum_{k=1}^{N} r_k\phi_k(t)}_{} + n_0(t) \qquad \text{where } r_k = s_{mk} + n_k \end{aligned}$$

Projection of r(t) on N-dim signal space

Graphical Illustration

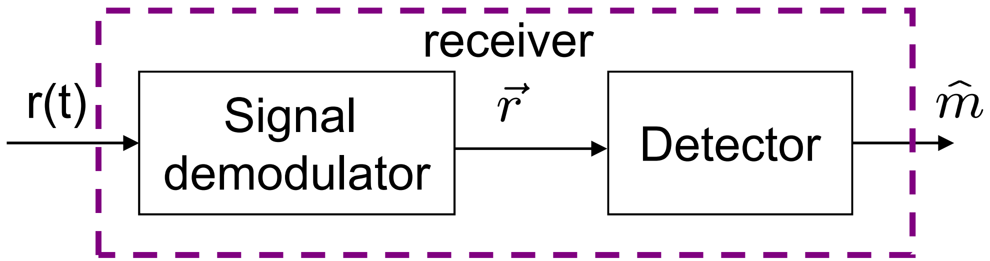
- In vector forms, we have

$$\vec{r} = \vec{s}_i + \vec{n}$$



Receiver Structure

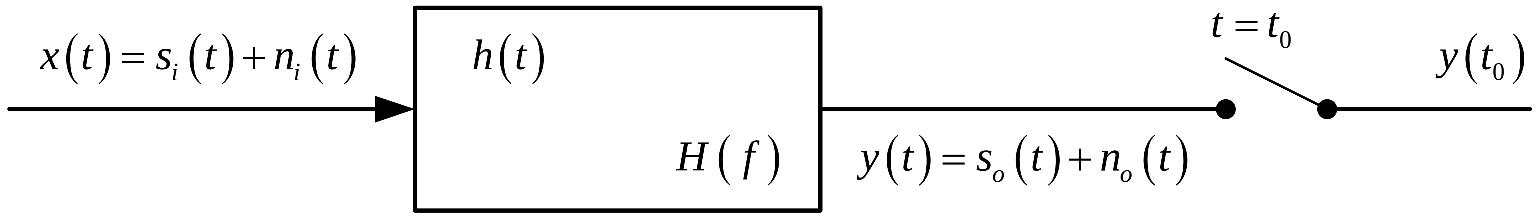
- Subdivide the receiver into two parts
  - Signal demodulator: to convert the received waveform r(t) into an N-dim vector $\vec{r} = (r_1, r_2, \ldots, r_N)$
  - Detector: to decide which of the M possible signal waveforms was transmitted based on observation of the vector $\vec{r}$



- Two realizations of the signal demodulator
  - Correlation-Type demodulator
  - Matched-Filter-Type demodulator

7.3 What is Matched Filter?

- The matched filter (MF) is the optimal linear filter for maximizing the output SNR.
- Derivation of the MF



  - Input signal component $s_i(t) \leftrightarrow A(f) = \int_{-\infty}^{\infty} s_i(t) e^{-j\omega t} dt$
  - Input noise component $n_i(t)$ with PDS $S_{n_i}(f) = N_0/2$
  - The signal component in the filter output is

$$s_o(t) = \int_{-\infty}^{\infty} s_i(t-\tau) h(\tau) d\tau$$

$$= \int_{-\infty}^{\infty} A(f) H(f) e^{j\omega t} df$$

Output SNR

- At the sampling instance $t = t_0$, $s_o(t_0) = \int_{-\infty}^{\infty} A(f)H(f)e^{j\omega t_0} df$
- Average power of the output noise is

$$N = E\{n_o^2(t)\} = \frac{N_0}{2}\int_{-\infty}^{\infty} |H(f)|^2 df$$

- Now the problem is to select the filter's freq. response that maximizes the output SNR, defined as

$$\boxed{d = \frac{s_o^2(t_0)}{E\{n_o^2(t)\}} = \frac{\left[\int_{-\infty}^{\infty} A(f)H(f)e^{j\omega t_0} df\right]^2}{\frac{N_0}{2}\int_{-\infty}^{\infty} |H(f)|^2 df}}$$

Find **H(f) that can maximize d**

Maximum Output SNR

- Schwarz's inequality

$$\int_{-\infty}^{\infty}|F(x)|^2 dx \int_{-\infty}^{\infty}|Q(x)|^2 dx \geq \left|\int_{-\infty}^{\infty} F^*(x) Q(x) dx\right|^2$$

with equality holds when $F(x) = CQ(x)$ for any arbitrary constant C.

- Let $\begin{cases} F^*(x) = A(f)e^{j\omega t_0} \\ Q(f) = H(f) \end{cases}$, then

$$\boxed{d \leq \frac{\int_{-\infty}^{\infty}|A(f)|^2 df \int_{-\infty}^{\infty}|H(f)|^2 df}{\frac{N_0}{2}\int_{-\infty}^{\infty}|H(f)|^2 df} = \frac{\int_{-\infty}^{\infty}|A(f)|^2 df}{\frac{N_0}{2}} = \frac{2E}{N_0}}$$

E: signal energy

Solution of Matched Filter

- When the max output SNR $2E/N_0$ is achieved, we have

$$\boxed{\begin{array}{c} H_m(f) = A^*(f)e^{-j\omega t_0} \\ \updownarrow \\ h_m(t) = s_i(t_0 - t) \end{array}} \longleftarrow$$

$$\begin{aligned} h_m(t) &= \int_{-\infty}^{\infty} H_m(f)e^{j\omega t}df \\ &= \int_{-\infty}^{\infty} A^*(f)e^{-j\omega(t_0-t)}df \\ &= s_i^*(t_0 - t) \end{aligned}$$

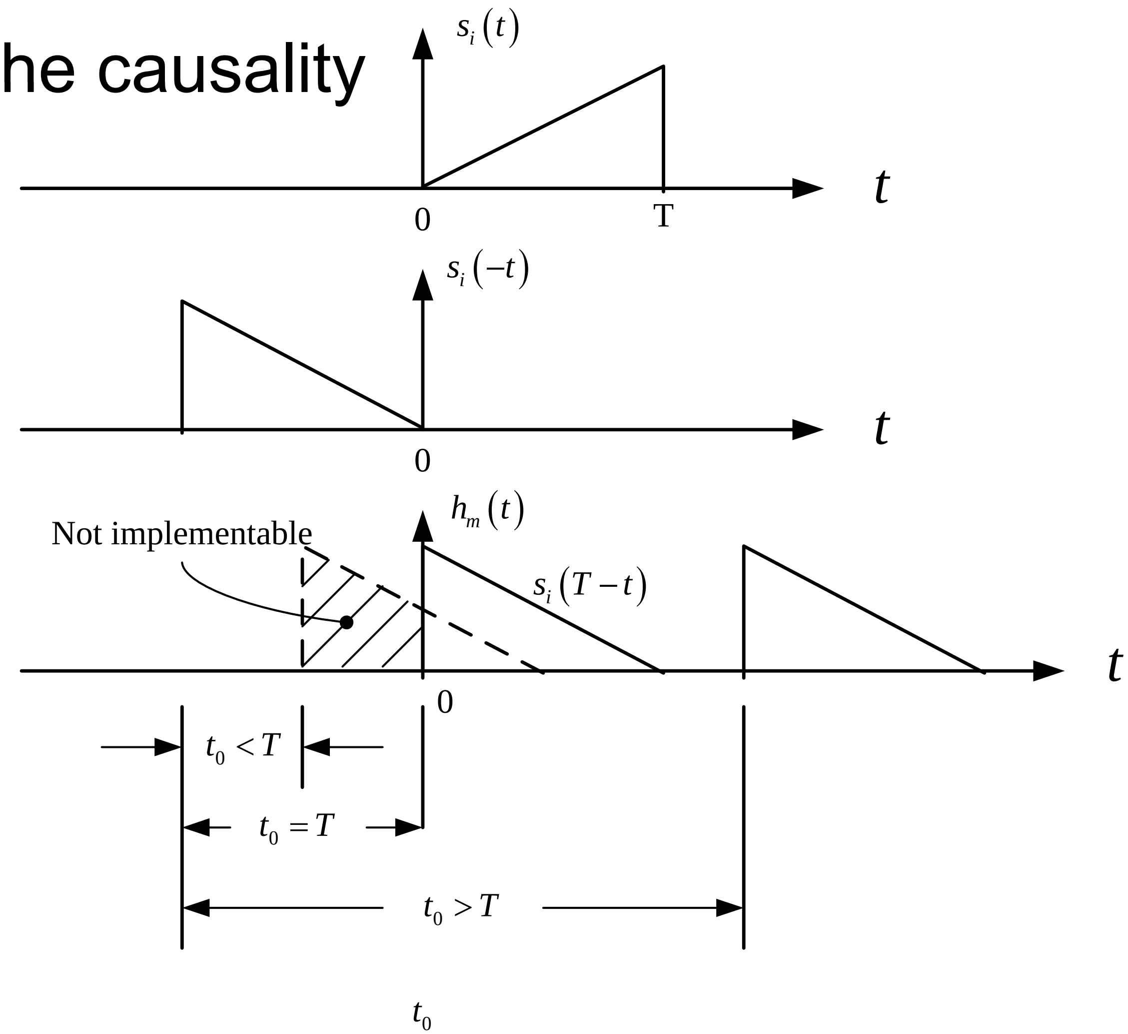
  - The transfer function of MF is the complex conjugate of the input signal spectrum
  - The impulse response of MF is a time-reversal and delayed version of the input signal s(t)

Properties of MF (1)

- Choice of t_0 versus the causality

$$h_m(t) = \begin{cases} s_i(t_0 - t) & 0 \le t < t_0 \\ 0 & \text{otherwise} \end{cases}$$

where $\quad t_0 \ge T$



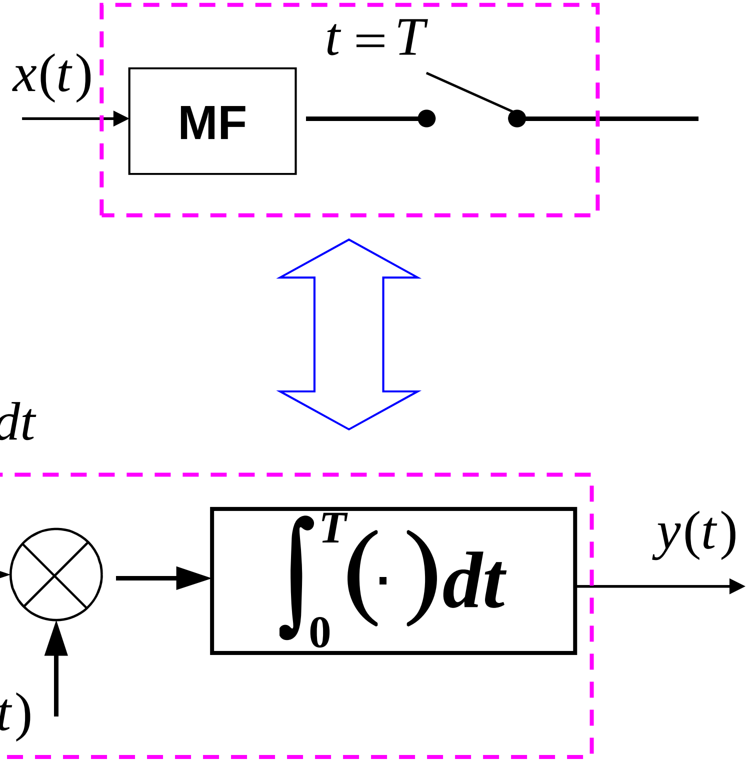
Properties of MF (2)

- Equivalent form – Correlator
  - Let $s_i(t)$ is within [0, T]

$$y(t) = s_o(t) + n_o(t) = \left[s_i(t) + n_i(t)\right] * h_m(t)$$
$$= \int_0^T \left[s_i(\tau) + n_i(\tau)\right] s_i(T - t + \tau)\, d\tau$$

  - Observe at sampling time $t = T$

$$y(T) = \int_0^T \left[s_i(\tau) + n_i(\tau)\right] s_i(\tau)\, d\tau = \int_0^T x(t)\, s_i(t)\, dt$$



Correlation integration

Correlation Integration

- Correlation function in time domain

$$R_{12}(\tau)=\int_{-\infty}^{\infty}s_1(t)s_2(t+\tau)dt=\int_{-\infty}^{\infty}s_1(t-\tau)s_2(t)dt=R_{21}(-\tau)$$

- Autocorrelation function $R(\tau)=\int_{-\infty}^{\infty}s(t)s(t+\tau)dt$
  - $R(\tau)=R(-\tau)$
  - $R(0)\geq R(\tau)$
  - $R(0)=\int_{-\infty}^{\infty}s^2(t)dt=E$
  - $R(\tau)\leftrightarrow|A(f)|^2 \qquad R(0)=\int_{-\infty}^{\infty}s^2(t)dt=\int_{-\infty}^{\infty}|A(f)|^2df$

Properties of MF (3)

- MF output signal is the autocorre. function of input signal

$$s_o(t) = \int_{-\infty}^{\infty} s_i(t-u) h_m(u) du = \int_{-\infty}^{\infty} s_i(t-u) s_i(t_0 - u) du$$

$$= \int_{-\infty}^{\infty} s_i(\mu) s_i[\mu + t - t_0] d\mu = R_{s_0}(t - t_0)$$

  - The peak value of $s_0(t)$ happens at $t = t_0$

$$s_o(t_0) = \int_{-\infty}^{\infty} s_i^2(\mu) du = E$$

  - $s_0(t)$ is symmetric at $t = t_0$

$$A_o(f) = A(f) H_m(f) = |A(f)|^2 e^{-j\omega t_0}$$

$$s_o(t) = s_0{}'(t - t_0)$$

Properties of MF (4)

- MF output noise
  - The statistical autocorrelation of $n_0(t)$ depends on the autocorrelation of $s_i(t)$

$$R_{n_o}(\tau) = E\{n_o(t)n_o(t+\tau)\} = \frac{N_0}{2}\int_{-\infty}^{\infty} h_m(u)h_m(u+\tau)du$$

$$= \frac{N_0}{2}\int_{-\infty}^{\infty} s_i(t)s_i(t-\tau)dt$$

  - Average power

$$E\{n_o^2(t)\} = R_{n_o}(0) = \frac{N_0}{2}\int_{-\infty}^{\infty} s_i^2(\mu)du \quad \text{(time domain)}$$

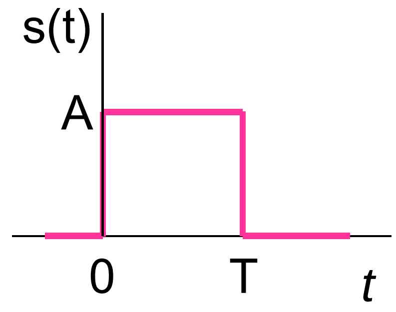
$$= \frac{N_0}{2}\int_{-\infty}^{\infty} |A(f)|^2 df = \frac{N_0}{2}\int_{-\infty}^{\infty} |H_m(f)|^2 df \quad \text{(freq. domain)}$$

$$= \frac{N_0}{2}E$$

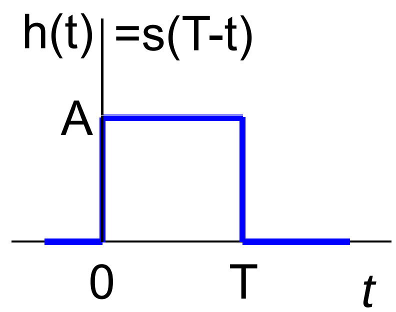
Example: MF for a rectangular pulse

- Consider a rectangular pulse s(t)

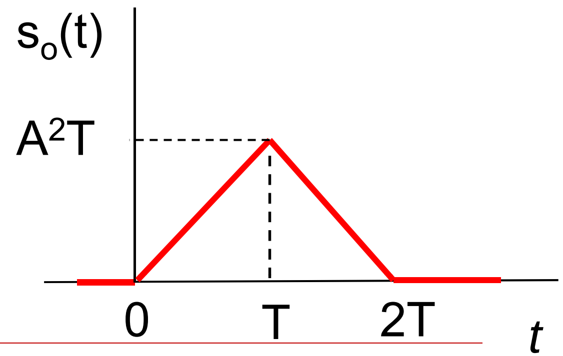
$$E_s = A^2 T$$



- The impulse response of a filter matched to s(t) is also a rectangular pulse



- The output of the matched filter $s_o(t)$ is $h(t) * s(t)$



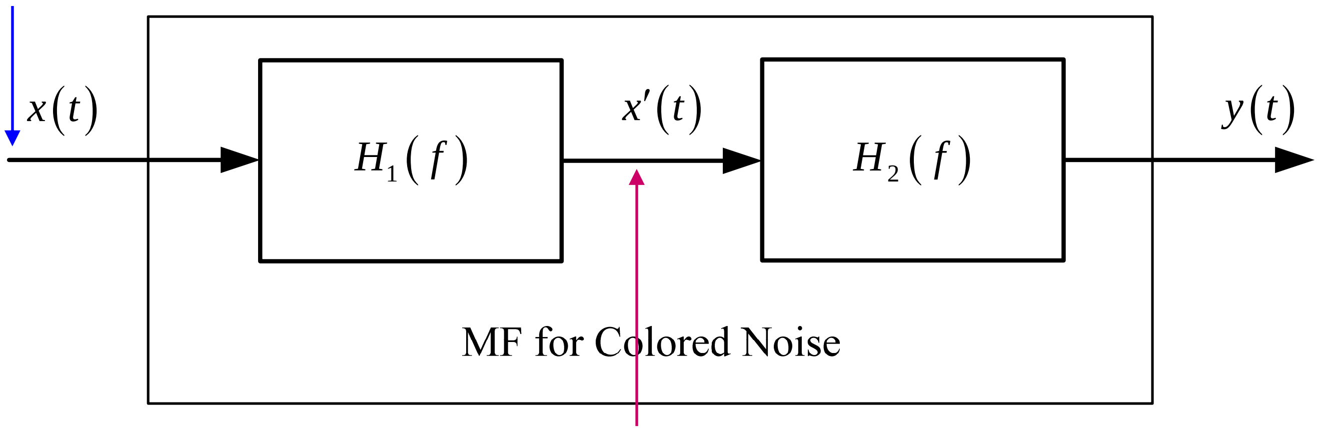
- The output SNR is

$$(SNR)_o = \frac{2}{N_0}\int_0^T s^2(t)dt = \frac{2A^2T}{N_0}$$

What if the noise is Colored?

- Basic idea: preprocess the combined signal and noise such that the non-white noise becomes white noise - Whitening Process

$x(t) = s_i(t) + n(t)$ **where n(t) is colored noise with PSD** $S_n(f)$



$$x'(t) = s'(t) + n'(t)$$

Choose $H_1(f)$ so that n'(t) is white, i.e. $S_n'(f) = |H_1(f)|^2 S_n(f) = C$

$H_1(f)$, $H_2(f)$

- $H_1(f)$: $|H_1(f)|^2 = \dfrac{C}{S_n(f)}$
- H2(f) should match with s’(t) $A'(f) = H_1(f)A(f)$

$$H_2(f) = A'^*(f)e^{-j2\pi f t_0} = H_1^*(f)A^*(f)e^{-j2\pi f t_0}$$

- The overall transfer function of the cascaded system is

$$H(f) = H_1(f)\cdot H_2(f) = H_1(f)H_1^*(f)A^*(f)e^{-j2\pi f t_0}$$

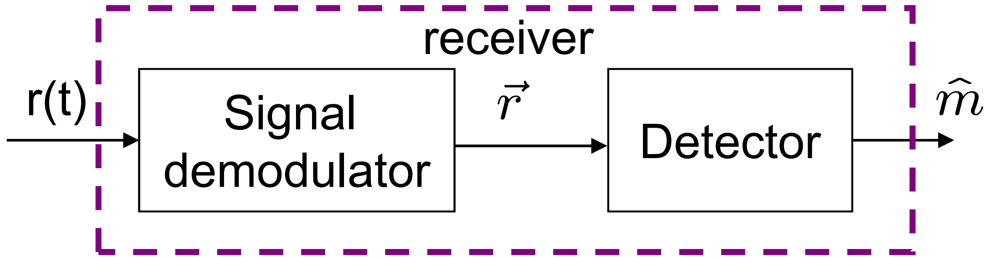
$$= |H_1(f)|^2 A^*(f)e^{-j2\pi f t_0}$$

$$\boxed{H(f) = \frac{A^*(f)}{S_n(f)}e^{-j2\pi f t_0}}$$

MF for colored noise

Update

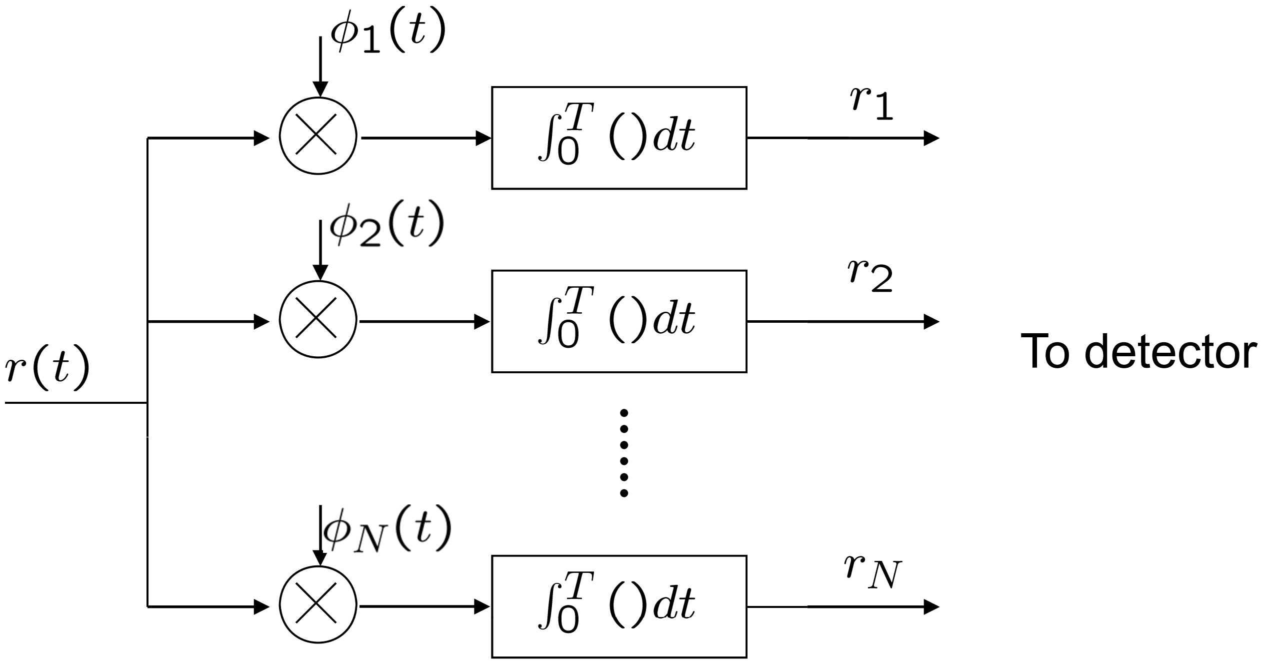
- We have discussed what is matched filter
- Let us now come back to the optimal receiver structure



- Two realizations of the signal demodulator
  - Correlation-Type demodulator
  - Matched-Filter-Type demodulator

Correlation Type Demodulator

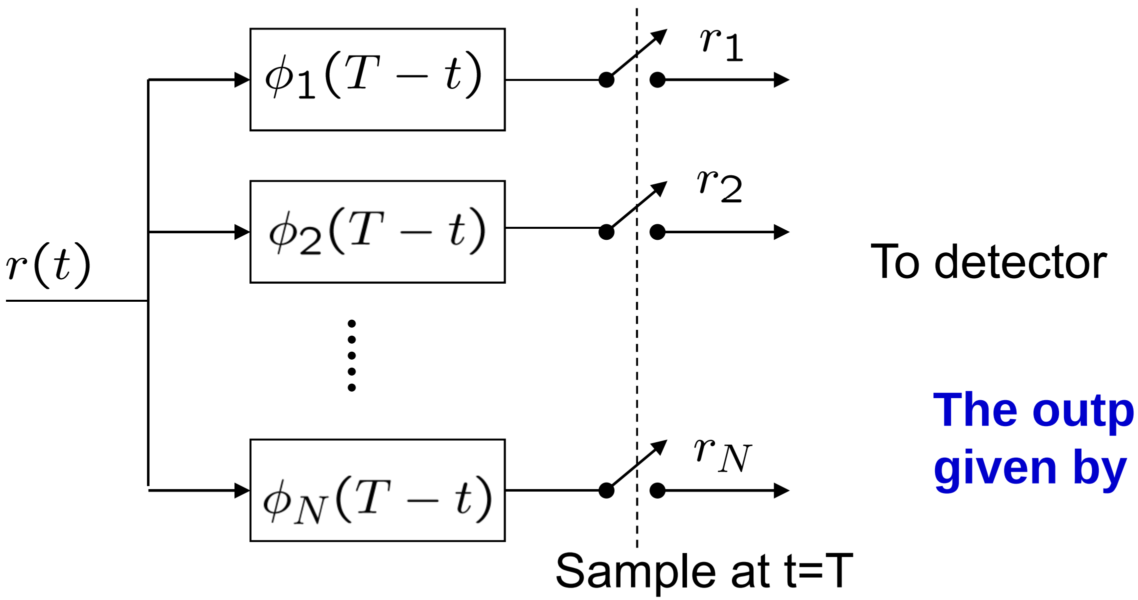
- The received signal r(t) is passed through a parallel bank of N cross correlators which basically compute the projection of r(t) onto the N basis functions $\{\phi_k(t), k = 1, \ldots N\}$



Matched-Filter Type Demodulator

- Alternatively, we may apply the received signal r(t) to a bank of N matched filters and sample the output of filters at t = T. The impulse responses of the filters are

$$h_k(t) = \phi_k(T-t), \quad 0 \le t \le T$$



The output SNR is the maximum, given by

$$SNR_o = \frac{2E_s}{N_0}$$

- We have demonstrated that
  - for a signal transmitted over an AWGN channel, either a correlation type demodulator or a matched filter type demodulator produces the vector $\vec{r} = (r_1, r_2, \ldots, r_N)$ which contains all the necessary information in r(t)
- Now, we will discuss
  - the design of a signal detector that makes a decision of the transmitted signal in each signal interval based on the observation of $\vec{r}$, such that the probability of making an error is minimized (or correct probability is maximized)

Decision Rules

Recall that

- MAP decision rule:

choose $\hat{m} = m_k$ if and only if

$$P_k f(\vec{r}|m_k) > P_i f(\vec{r}|m_i); \text{ for all } i \neq k$$

- ML decision rule

choose $\hat{m} = m_k$ if and only if

$$f(\vec{r}|m_k) > f(\vec{r}|m_i); \text{ for all } i \neq k$$

In order to apply the MAP or ML rules, we need to evaluate the likelihood function $\boldsymbol{f(\vec{r}|m_k)}$

Distribution of the Noise Vector

- Since $n_w(t)$ is a Gaussian random process,
  - $n_k = \int_0^T n_w(t)\phi_k(t)dt$ is a Gaussian random variable (from definition)
- Mean: $E[n_k] = \int_0^T E[n_w(t)]\phi_k(t)dt = 0 \ , \ k = 1, \ldots, N$
- Correlation between n_j and n_k

$$\begin{aligned} E[n_j n_k] &= E\left[\int_0^T n_w(t)\phi_j(t)dt \cdot \int_0^T n_w(\tau)\phi_k(\tau)d\tau\right] \\ &= E\left[\int_0^T \int_0^T n_w(t)n_w(\tau)\phi_j(t)\phi_k(\tau)dtd\tau\right] \\ &= \int_0^T \int_0^T E[n_w(t)n_w(\tau)]\phi_j(t)\phi_k(\tau)dtd\tau \\ &= \int_0^T \int_0^T \frac{N_0}{2}\delta(t-\tau)\phi_j(t)\phi_k(\tau)dtd\tau \end{aligned}$$

PSD of $n_w(t)$ is

$S_n(f) = N_0/2$

- Using the property of a delta function $\int_{\infty}^{\infty} g(t)\delta(t-a)dt = g(a)$ we have:

$$E[n_j n_k] = \frac{N_0}{2}\int_0^T \phi_j(\tau)\phi_k(\tau)d\tau \ = \begin{cases} \frac{N_0}{2}, & j = k \\ 0, & j \neq k \end{cases}$$

- Therefore, n_j and n_k ($j \neq k$) are uncorrelated Gaussian random variables
  - They are independent with zero-mean and variance $N_0/2$
- The joint pdf of $\vec{n} = (n_1, \ldots, n_N)$

$$\begin{aligned} p(n_1, \ldots, n_N) &= \prod_{k=1}^{N} p(n_k) = \prod_{k=1}^{N} \frac{1}{\sqrt{\pi N_0}} \exp\left(-n_k^2/N_0\right) \\ &= (\pi N_0)^{-N/2} \exp\left(-\sum_{k=1}^{N} n_k^2/N_0\right) \end{aligned}$$

Likelihood Function

- If m_k is transmitted, $\vec{r} = \vec{s}_k + \vec{n}$ with $r_j = s_{kj} + n_j$
- $E[r_j|m_k] = s_{kj} + E[n_j] = s_{kj}$
- $Var[r_j|m_k] = Var[n_j] = N_0/2$
- Transmitted signal values in each dimension represent the mean values for each received signal
- Conditional pdf of the random variables $\vec{r} = (r_1, r_2, \ldots, r_N)$

$$f(\vec{r}|m_k) = \prod_{j=1}^{N} \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{(r_j - s_{kj})^2}{N_0}\right)$$

$$= (\pi N_0)^{-N/2} \exp\left(-\frac{\sum_{j=1}^{N}(r_j - s_{kj})^2}{N_0}\right)$$

Log-Likelihood Function

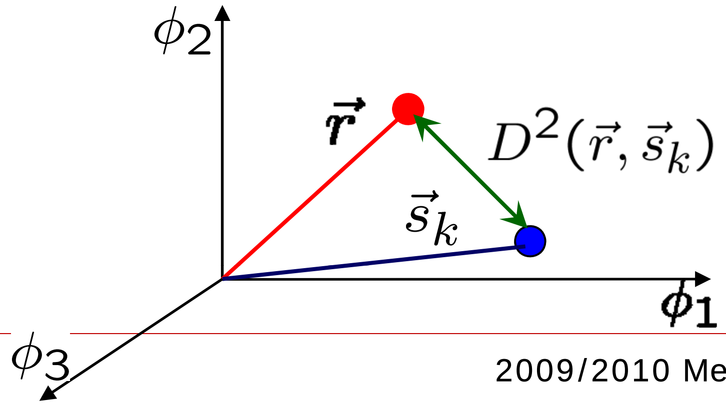
- To simplify the computation, we take the natural logarithm of $f(\vec{r}|m_k)$, which is a monotonic function. Thus

$$\ln f(\vec{r}|m_k) = -\frac{N}{2}\ln(\pi N_0) - \frac{1}{N_0}\sum_{j=1}^{N}(r_j - s_{kj})^2$$

- Let

$$D^2(\vec{r}, \vec{s}_k) = \sum_{j=1}^{N}(r_j - s_{k,j})^2 \quad = \|\vec{r} - \vec{s}_k\|^2$$

- $D(\vec{r}, \vec{s}_k)$ is the Euclidean distance between $\vec{r}$ and $\vec{s}_k$ in the N-dim signal space. It is also called distance metrics



Optimum Detector

- MAP rule:

$$\hat{m} = \arg \max_{\{m_1,\ldots,m_M\}} f(\vec{r}|m_k)P(m_k)$$

$$= \arg \max_{\{m_1,\ldots,m_M\}} \ln\left[f(\vec{r}|m_k)P(m_k)\right]$$

$$= \arg \max_{\{m_1,\ldots,m_M\}} \left\{-\frac{1}{N_0}\|\vec{r}-\vec{s}_k\|^2 + \ln P_k\right\}$$

$$\boxed{= \arg \min_{\{m_1,\ldots,m_M\}} \left\{\|\vec{r}-\vec{s}_k\|^2 - N_0 \ln P_k\right\}}$$

- ML rule:

$$\boxed{\hat{m} = \arg \min_{\{m_1,\ldots,m_M\}} \|\vec{r}-\vec{s}_k\|^2}$$

ML detector chooses $\hat{m} = m_k$ iff received vector $\vec{r}$ is closer to $\vec{s}_k$ in terms of Euclidean distance than to any other $\vec{s}_i$ for i ≠ k

$\Rightarrow$ Minimum distance detection

(will discuss more in decision region)

Optimal Receiver Structure

- From previous expression we can develop a receiver structure using the following derivation

$$-\sum_{j=1}^{N}(r_j - s_{kj})^2 + N_0 \ln P_k = -\sum_{j=1}^{N} r_j^2 - \sum_{j=1}^{N} s_{kj}^2 + 2\sum_{j=1}^{N} r_j s_{kj} + N_0 \ln P_k$$

$$= -\|\vec{r}\|^2 - \|\vec{s}_k\|^2 + 2\vec{r}\cdot\vec{s}_k + N_0 \ln P_k$$

in which

$\|\vec{s}_k\|^2 = \int_0^T s_k^2(t)dt = E_k$ = signal energy

$\vec{r}\cdot\vec{s}_k = \int_0^T s_k(t)r(t)dt$ = correlation between the received signal vector and the transmitted signal vector

$\|\vec{r}\|^2$ = common to all M decisions and hence can be ignored

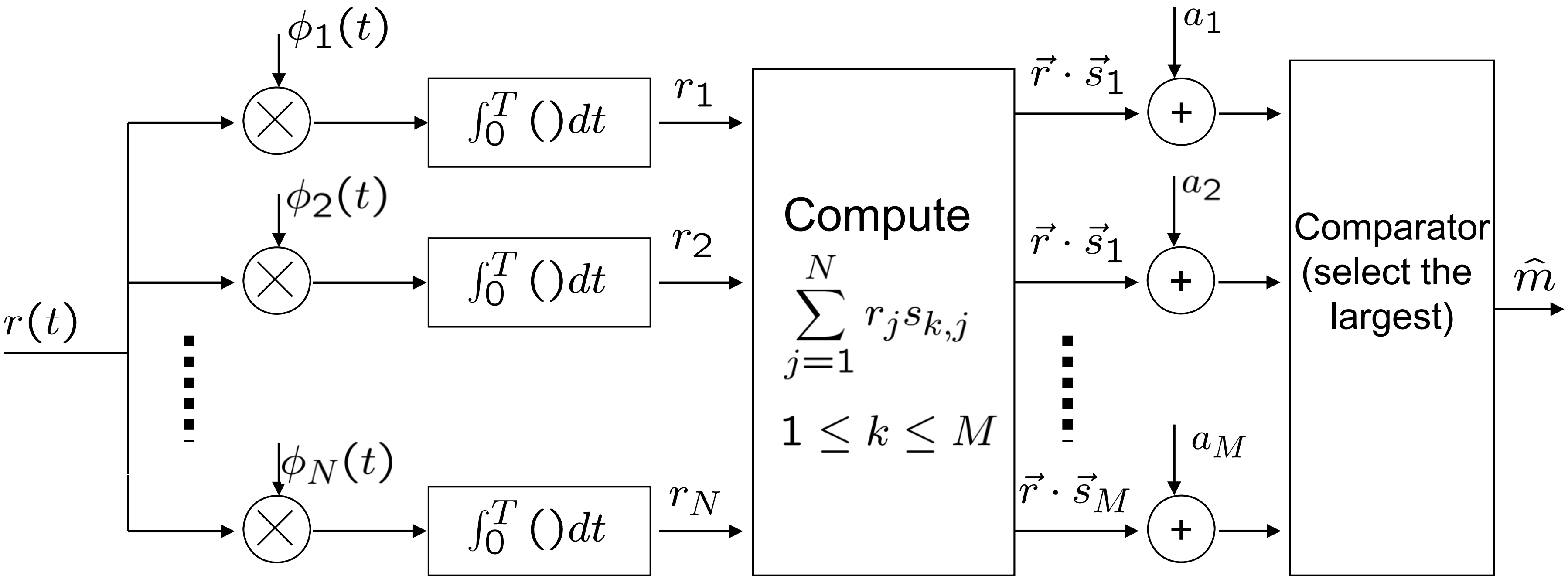
- The new decision function becomes

$$\hat{m} = \arg \max_{m_1,\ldots,m_M} \left\{ \vec{r} \cdot \vec{s}_k - \frac{E_k}{2} + \frac{N_0}{2} \ln P_k \right\}$$

- Now we are ready draw the implementation diagram of MAP receiver (signal demodulator + detector)

MAP Receiver Structure

Method 1 (Signal Demodulator + Detector)

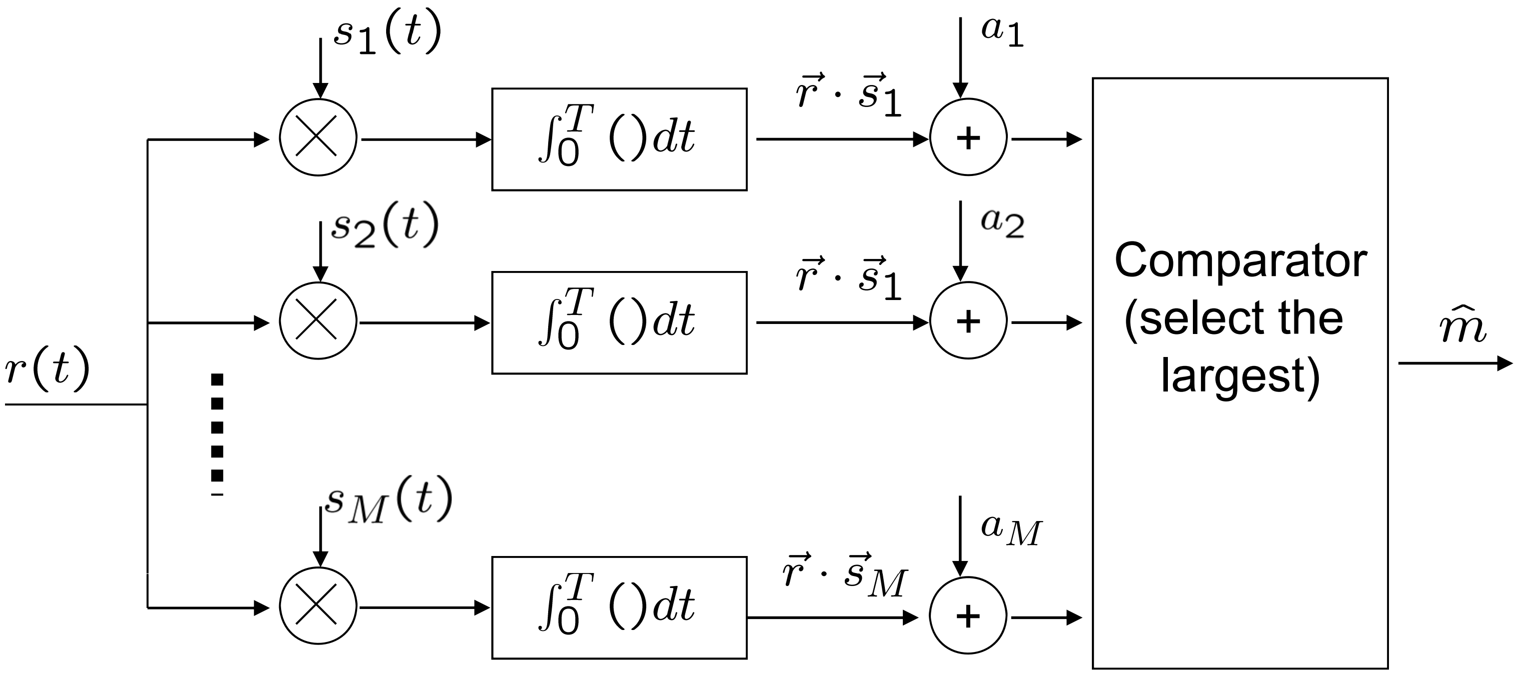


$$a_k = \frac{N_0}{2} \ln p_k - \frac{E_k}{2}$$

This part can also be implemented using matched filters

MAP Receiver Structure

Method 2 (Integrated demodulator and detector)



$$a_k = \frac{N_0}{2} \ln p_k - \frac{E_k}{2}$$

This part can also be implemented using matched filters

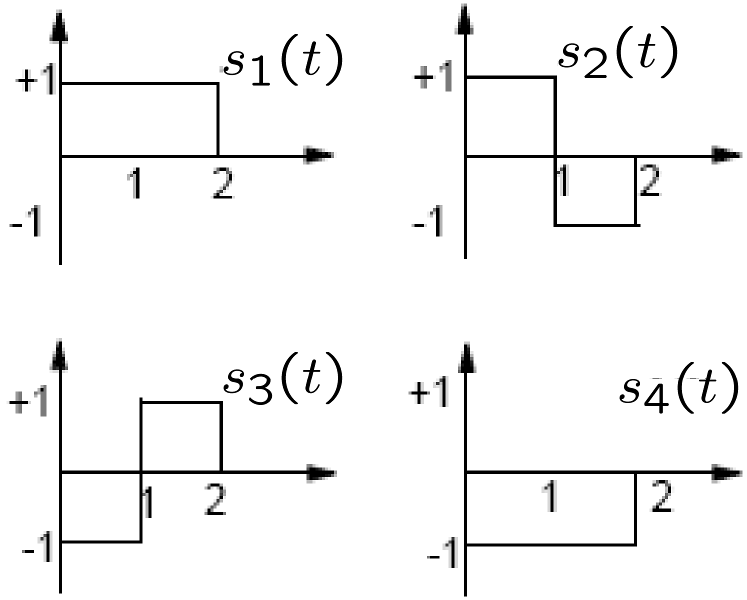
$$\hat{m} = \arg \max_{m_1,\ldots,m_M} \left\{ \vec{r} \cdot \vec{s}_k - \frac{E_k}{2} + \frac{N_0}{2} \ln P_k \right\}$$

Method 1 vs. Method 2

- Both receivers perform identically
- Choice depends on circumstances
- For instance, if N < M and $\{\phi_j(t)\}$ are easier to generate than $\{s_k(t)\}$, then the choice is obvious

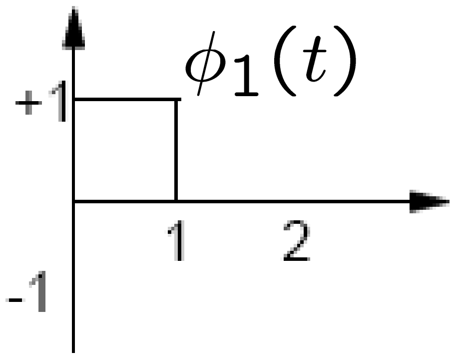
Example: optimal receiver design

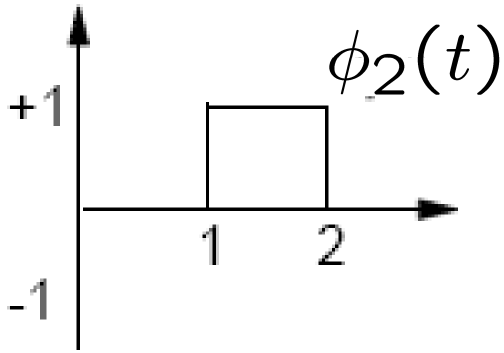
- Consider the signal set



Example (cont'd)

- Suppose we use the following basis functions





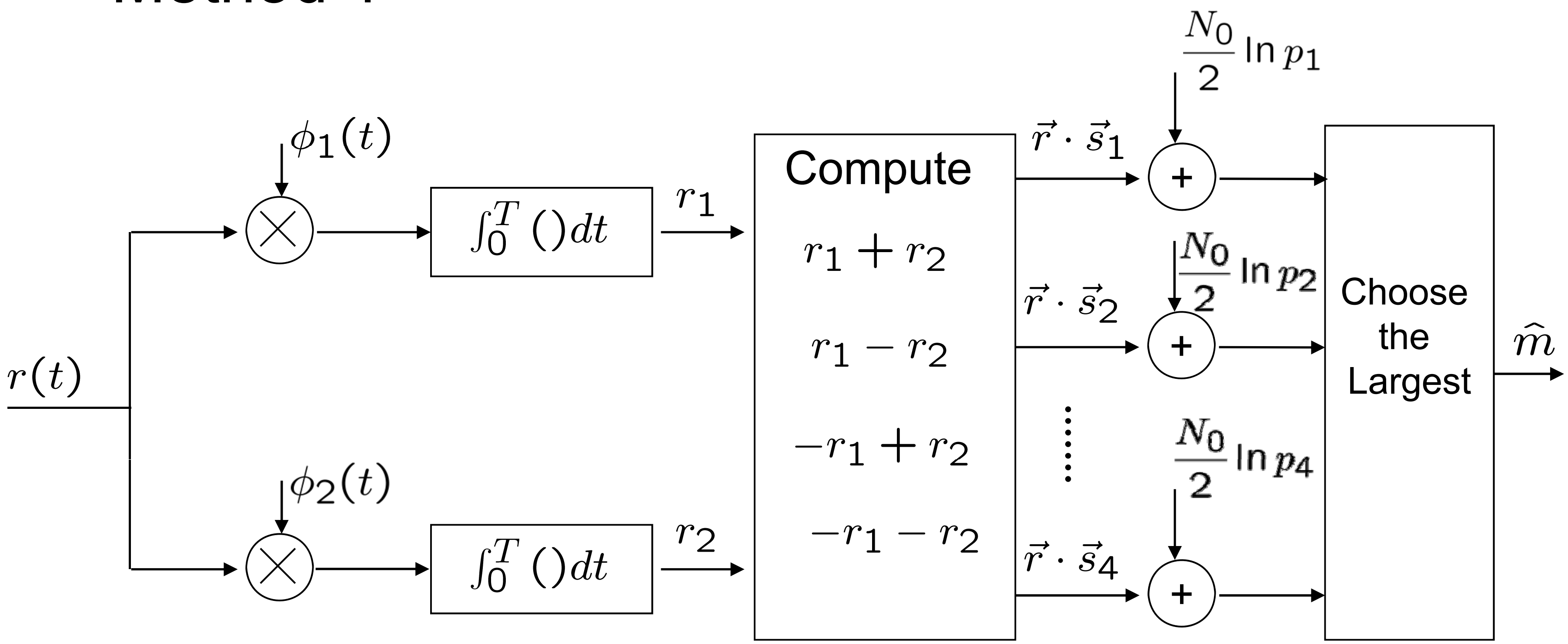
$s_1(t) = 1\,\phi_1(t) + 1\cdot\phi_2(t)$ $\qquad$ $s_2(t) = 1\,\phi_1(t) - 1\cdot\phi_2(t)$

$s_3(t) = -1\,\phi_1(t) + 1\cdot\phi_2(t)$ $\qquad$ $s_4(t) = -1\,\phi_1(t) - 1\cdot\phi_2(t)$

- Since the energy is the same for all four signals, we can drop the energy term from $a_k = \frac{N_0}{2}\ln p_k$

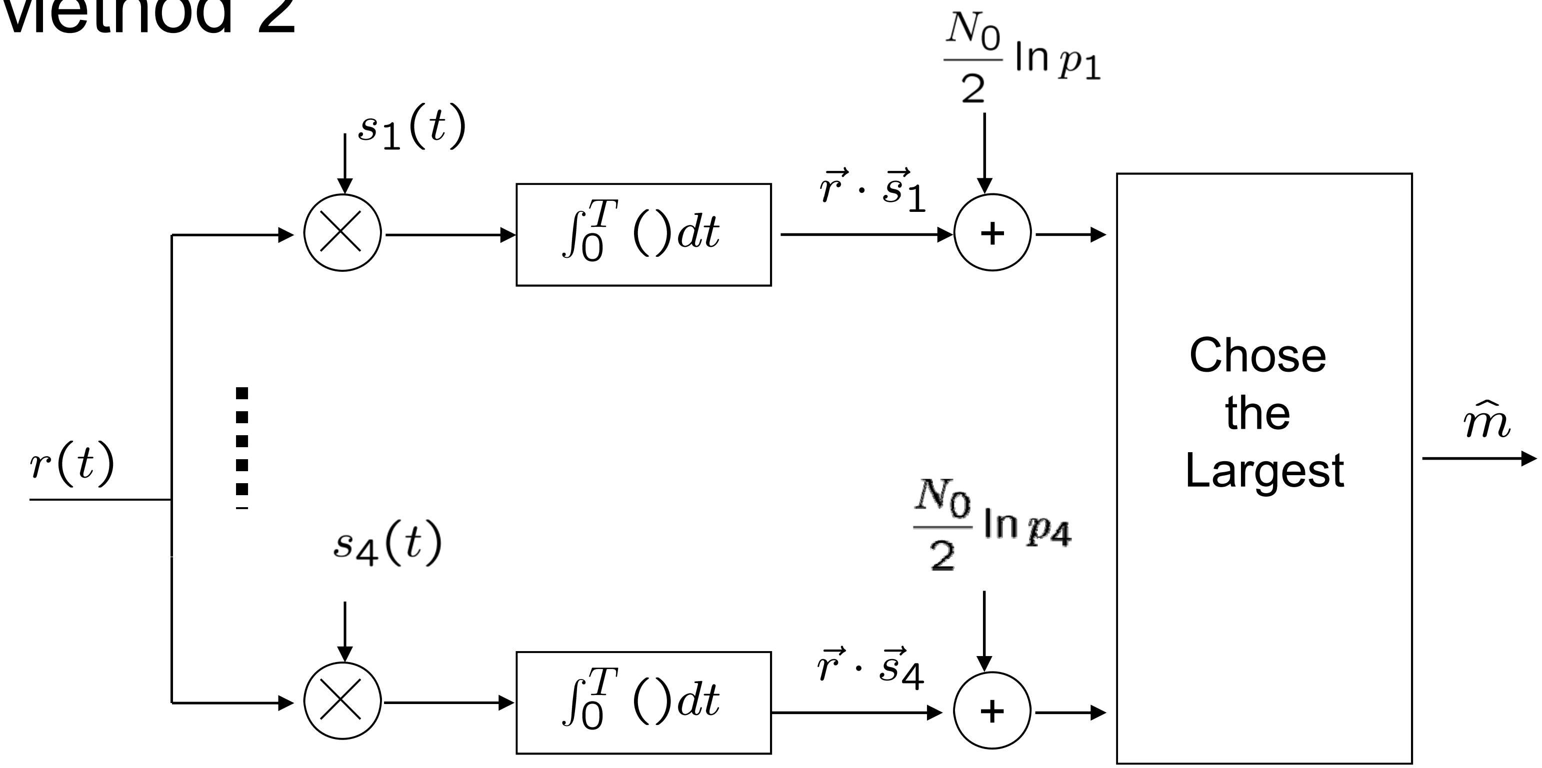
Example (cont'd)

- Method 1



Example (cont'd)

- Method 2



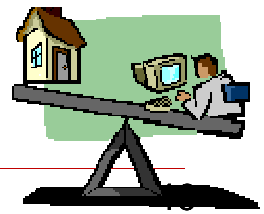
Exercise

In an additive white Gaussian noise channel with a noise power-spectral density of $N_0/2$, two equiprobable messages are transmitted by

$$s_1(t) = \begin{cases} \dfrac{At}{T} & 0 \le t \le T \\ 0 & \text{otherwise} \end{cases}$$

$$s_2(t) = \begin{cases} A - \dfrac{At}{T} & 0 \le t \le T \\ 0 & \text{otherwise} \end{cases}$$

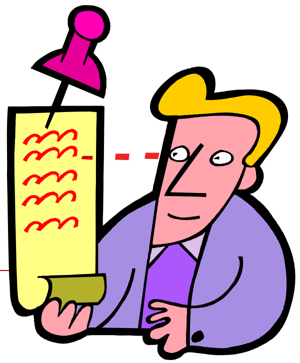
- Determine the structure of the optimal receiver.



Notes on Optimal Receiver Design

- The receiver is general for any signal forms

- Simplifications are possible under certain scenarios



- We have considered
  - MAP and ML decision rules
  - Correlation-type demodulator
  - Matched-filter-type demodulator
  - Implementation of optimal receiver

- We will now consider
  - Graphical interpretation of design regions
  - Analysis of probability of error
  - Union bound

7.4 Graphical Interpretation
– Decision Regions

- Signal space can be divided into M disjoint decision regions R_1 R_2, …, R_M.

If $\vec{r} \in R_k$ $\Rightarrow$ decide m_k was transmitted

Select decision regions so that P_e is minimized

- Recall that the optimal receiver sets $\hat{m} = m_k$ iff

  $\|\vec{r} - \vec{s}_k\|^2 - N_0 \ln P_k$ is minimized

- For simplicity, if one assumes p_k = 1/M, for all k, then the optimal receiver sets $\hat{m} = m_k$ iff

  $\|\vec{r} - \vec{s}_k\|^2$ is minimized

Decision Regions

- Geometrically, this means
  - Take projection of r(t) in the signal space (i.e. $\vec{r}$). Then, decision is made in favor of signal that is the closest to $\vec{r}$ in the sense of minimum Euclidean distance
  - And those observation vectors $\vec{r}$ with $\|\vec{r}-\vec{s}_k\|^2 < \|\vec{r}-\vec{s}_i\|^2$

  for all $i \neq k$ should be assigned to decision region R_k

Example: Binary Case

- Consider binary data transmission over AWGN channel with PSD $S_n(f) = N_0/2$ using

$$s_1(t) = -s_2(t) = \sqrt{E}\phi(t)$$

- Assume $P(m_1) \neq P(m_2)$
- Determine the optimal receiver (and optimal decision regions)

Solution

- Optimal decision making

$$\|\vec{r}-\vec{s}_1\|^2 - N_0 \ln P(m_1) \underset{\text{Choose } m_2}{\overset{\text{Choose } m_1}{\lessgtr}} \|\vec{r}-\vec{s}_2\|^2 - N_0 \ln P(m_2)$$

- Let $d_1 = \|\vec{r}-\vec{s}_1\|$ and $d_2 = \|\vec{r}-\vec{s}_2\|$

- Equivalently,

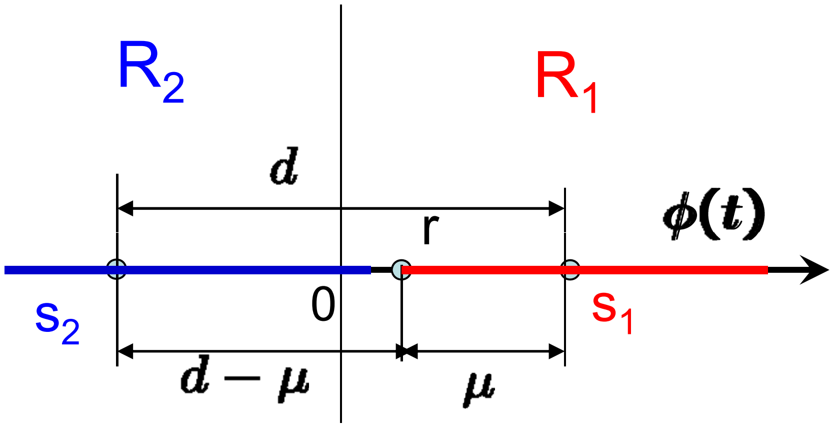
$$d_1^2 - d_2^2 \underset{\text{Choose } m_2}{\overset{\text{Choose } m_1}{\lessgtr}} \underbrace{N_0 \ln \frac{P(m_1)}{P(m_2)}}_{\text{Constant } c}$$

R_1: $d_1^2 - d_2^2 < c$ **and** R_2: $d_1^2 - d_2^2 > c$

Solution (cont'd)

- Now consider the example with $\vec{r}$ on the decision boundary

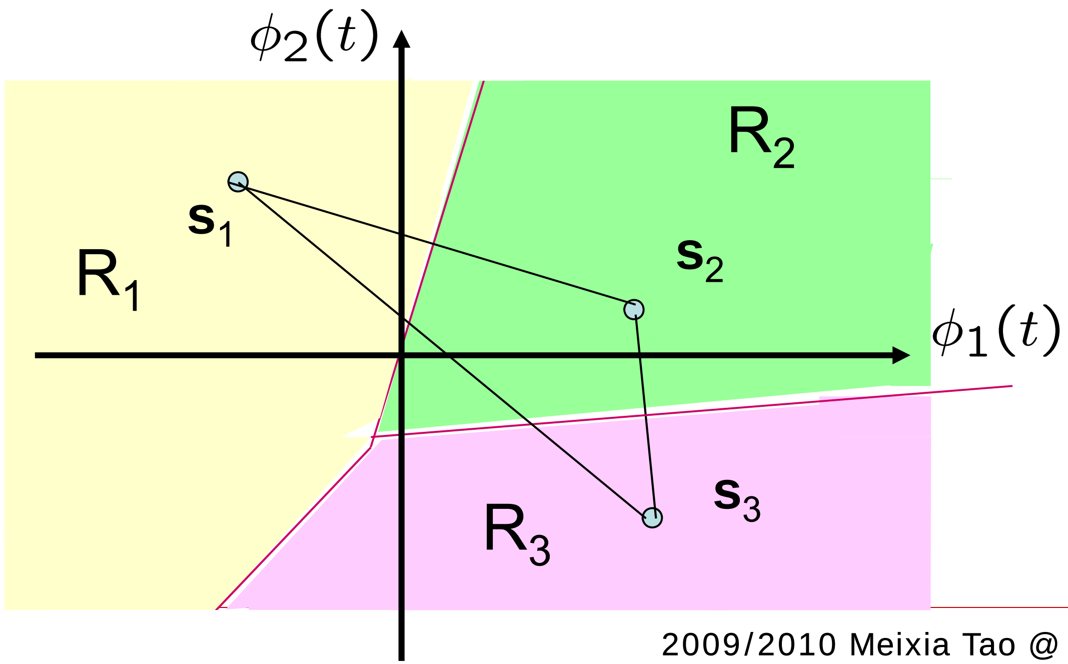
$$\begin{cases} d = d_1 + d_2 \\ d_1^2 = \mu^2 \\ d_2^2 = (d-\mu)^2 \end{cases} \Rightarrow \begin{aligned} & d_1^2 - d_2^2 = 2d\mu - d^2 \equiv c \\ & \mu = \frac{c+d^2}{2d} = \frac{d}{2} + \frac{N_0}{2d} \ln \frac{P(m_1)}{P(m_2)} \end{aligned}$$



$$\mu \begin{cases} = d/2 & \text{if } P(m_1) = P(m_2) \\ > d/2 & \text{if } P(m_1) > P(m_2) \\ < d/2 & \text{if } P(m_1) < P(m_2) \end{cases}$$

Determining the Optimum Decision Regions

- In general, boundaries of decision regions are perpendicular bisectors of the lines joining the original transmitted signals
- Example: three equiprobable 2-dim signals



Example: Decision Region for QPSK

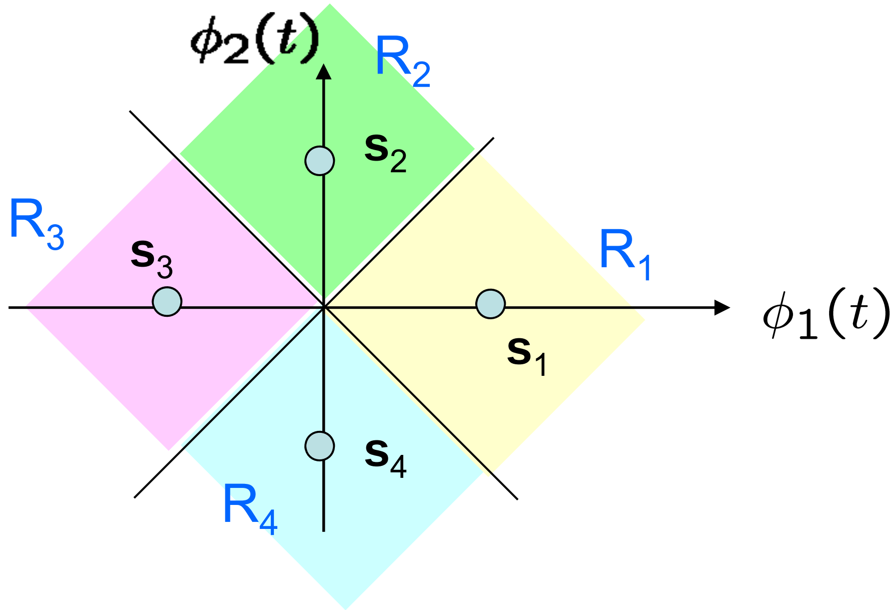
- Assume all signals are equally likely
- All 4 signals could be written as the linear combination of two basis functions
- Constellations of 4 signals

$\mathbf{s}_1=(1,0)$

$\mathbf{s}_2=(0,1)$

$\mathbf{s}_3=(-1,0)$

$\mathbf{s}_4=(0,-1)$



Exercise

Three equally probable messages m1, m2, and m3 are to be transmitted over an AWGN channel with noise power-spectral density $N_0/2$. The messages are

$$s_1(t)=\begin{cases}1 & 0\le t\le T\\ 0 & otherwise\end{cases}$$

$$s_2(t)=-s_3(t)=\begin{cases}1 & 0\le t\le \frac{T}{2}\\ -1 & \frac{T}{2}\le t\le T\\ 0 & otherwise\end{cases}$$

1. What is the dimensionality of the signal space ?
2. Find an appropriate basis for the signal space (Hint: You can find the basis without using the Gram-Schmidt procedure).
3. Draw the signal constellation for this problem.
4. Sketch the optimal decision regions R1, R2, and R3.

Notes on Decision Regions

- Boundaries are perpendicular to a line drawn between two signal points
- If signals are equiprobable, decision boundaries lie exactly halfway in between signal points
- If signal probabilities are unequal, the region of the less probable signal will shrink

7.5 Probability of Error using Decision Regions

- Suppose m_k is transmitted and $\vec{r}$ is received
- Correct decision is made when $\vec{r} \in R_k$ with probability

$$P(C|m_k) = P(\vec{r} \in R_k | m_k \text{ is sent})$$

- Averaging over all possible transmitted symbols, we obtain the average probability of making correct decision

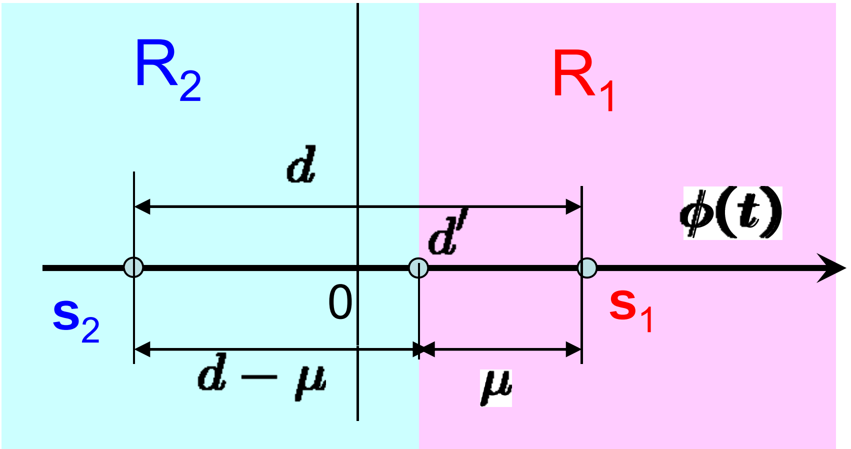
$$P(C) = \sum_{k=1}^{M} P(\vec{r} \in R_k | m_k \text{ is sent}) P(m_k)$$

- Average probability of error

$$\boxed{P_e = 1 - P(C) = 1 - \sum_{k=1}^{M} P(\vec{r} \in R_k | m_k \text{ is sent}) P(m_k)}$$

Example: P_e analysis

- Now consider our example with binary data transmission



$$\mu = \frac{d}{2} + \frac{N_0}{2d} \ln \frac{P(m_1)}{P(m_2)}$$

•Given m_1 is transmitted, then

$$\begin{aligned} P(C|s_1) &= P(r \in R_1|s_1) \\ &= P(s_1 + n > d') \\ &= P(n > -\mu) \end{aligned}$$

•Since n is Gaussian with zero mean and variance $N_0/2$

$$P(C|s_1) = 1 - Q\left(\frac{\mu}{\sqrt{N_0/2}}\right)$$

- Likewise

$$P(C|s_2) = P(s_2+n < d') = P(n < d-u) = 1-Q\left(\frac{d-\mu}{\sqrt{N_0/2}}\right)$$

- Thus,

$$P(C) = P(m_1)\left\{1 - Q\left[\frac{\mu}{\sqrt{N_0/2}}\right]\right\} + P(m_2)\left\{1 - Q\left[\frac{d-\mu}{\sqrt{N_0/2}}\right]\right\}$$

$$= 1 - P(m_1)Q\left[\frac{\mu}{\sqrt{N_0/2}}\right] - P(m_2)Q\left[\frac{d-\mu}{\sqrt{N_0/2}}\right]$$

$$\Rightarrow \quad P_e = P(m_1)Q\left[\frac{\mu}{\sqrt{N_0/2}}\right] + P(m_2)Q\left[\frac{d-\mu}{\sqrt{N_0/2}}\right]$$

where

$$d = 2\sqrt{E} \quad \text{and} \quad \mu = \frac{N_0}{4\sqrt{E}} \log\left[\frac{P(m_1)}{P(m_2)}\right] + \sqrt{E}$$

Example: P_e analysis (cont'd)

- Note that when $P(m_1) = P(m_2)$

$$\mu = \sqrt{E} = \frac{d}{2}$$

$$\Rightarrow \quad P_e = Q\left[\frac{d/2}{\sqrt{N_0/2}}\right] = Q\left[\sqrt{\frac{d^2}{2N_0}}\right] = Q\left[\sqrt{\frac{2E}{N_0}}\right]$$

$$= Q\left[\sqrt{\frac{E_1 + E_2 - 2\rho_{12}\sqrt{E_1 E_2}}{2N_0}}\right] = Q\left[\sqrt{\frac{2E}{N_0}}\right]$$

Example: P_e analysis (cont'd)

This example demonstrates an interesting fact:

- When optimal receiver is used, P_e does not depend upon the specific waveform used
- P_e depends only on their geometrical representation in signal space
- In particular, P_e depends on signal waveforms only through their energies (distance)

Exercise

Three equally probable messages m1, m2, and m3 are to be transmitted over an AWGN channel with noise power-spectral density $N_0/2$. The messages are

$$s_1(t) = \begin{cases} 1 & 0 \le t \le T \\ 0 & otherwise \end{cases}$$

$$s_2(t) = -s_3(t) = \begin{cases} 1 & 0 \le t \le \frac{T}{2} \\ -1 & \frac{T}{2} \le t \le T \\ 0 & otherwise \end{cases}$$

1. What is the dimensionality of the signal space ?
2. Find an appropriate basis for the signal space (Hint: You can find the basis without using the Gram-Schmidt procedure).
3. Draw the signal constellation for this problem.
4. Sketch the optimal decision regions R1, R2, and R3.
5. Which of the three messages is more vulnerable to errors and why ? In other words, which of $p(Error \mid m_i \quad transmitted), \quad i = 1, \quad 2, \quad 3$ is larger ?

General Expression for P_e

- Average probability of symbol error

$$P_e = 1 - P(C) = 1 - \sum_{k=1}^{M} P(\vec{r} \in R_k | m_k \text{ is sent}) P(m_k)$$

Likelihood function

- Since $$P(\vec{r} \in R_k | m_k \text{ is sent}) = \int_{R_k} f(\vec{r}|m_k) d\vec{r}$$

N-dim integration

- Thus we rewrite P_e in terms of likelihood functions, assuming that symbols are equally likely to be sent

$$P_e = 1 - \frac{1}{M} \sum_{k=1}^{M} \int_{R_k} f(\vec{r}|m_k) d\vec{r}$$

Union Bound

- Multi-dimension integrals are quite difficult to evaluate
- To overcome this difficulty, we resort to the use of bounds
- Now we develop a simple and yet useful upper bound for P_e, called union bound, as an approximation to the average probability of symbol error

Key Approximation

- Let A_{kj} denote the event that $\vec{r}$ is closer to $\vec{s}_j$ than to $\vec{s}_k$ in the signal space when m_k ($\vec{s}_k$) is sent
- Conditional probability of symbol error when m_k is sent

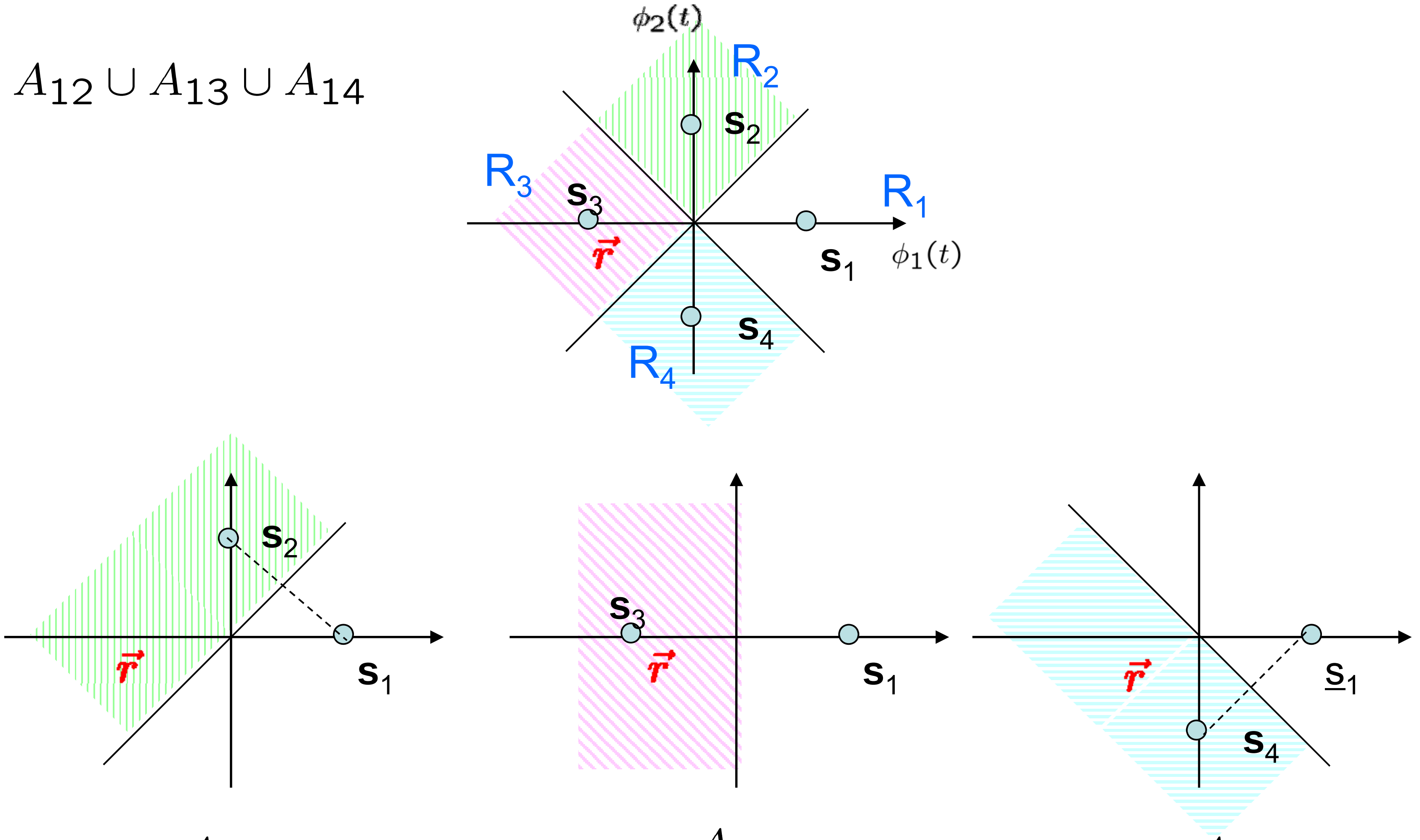
$$P(error|m_k) = P(\vec{r} \notin R_k|m_k) = P\left(\bigcup_{j\neq k} A_{kj}\right)$$

- But

$$P\left(\bigcup_{j\neq k} A_{kj}\right) \leq \sum_{\substack{j=1 \\ j \neq k}}^{M} P\left(A_{kj}\right)$$

Key Approximation (cont'd)

$A_{12} \cup A_{13} \cup A_{14}$



A_{12} A_{13} A_{14}

Pair-wise Error Probability

- Define the pair-wise (or component-wise) error probability as

$$P(\vec{s}_k \to \vec{s}_j) = P(A_{kj})$$

- It is equivalent to the probability of deciding in favor of $\vec{s}_j$ when $\vec{s}_k$ was sent in a simplified binary system that involves the use of two equally likely messages $\vec{s}_k$ and $\vec{s}_j$
- Then

$$P\left(\vec{s}_k \to \vec{s}_j\right) = P\left(n > d_{kj}/2\right) = Q\left(\sqrt{\frac{d_{kj}^2}{2N_0}}\right)$$

  - $d_{kj} = \|\vec{s}_k - \vec{s}_j\|$ is the Euclidean distance between $\vec{s}_k$ and $\vec{s}_j$

Union Bound

- Conditional error probability

$$P(error|m_k) \le \sum_{\substack{j=1 \\ j \neq k}}^{M} P(\vec{s}_k \to \vec{s}_j) = \sum_{\substack{j=1 \\ j \neq k}}^{M} Q\left(\sqrt{\frac{d_{kj}^2}{2N_0}}\right)$$

- Finally, with M equally likely messages, the average probability of symbol error is upper bounded by

$$\boxed{\begin{aligned} P_e &= \frac{1}{M}\sum_{k=1}^{M} P(error|m_k) \\ &\le \frac{1}{M}\sum_{k=1}^{M}\sum_{\substack{j=1 \\ j \neq k}}^{M} Q\left(\sqrt{\frac{d_{kj}^2}{2N_0}}\right) \end{aligned}}$$

$\Leftarrow$ The most general formulation of union bound

Union Bound (cont'd)

- Let $d_{\min}$ denote the minimum distance, i.e.

$$d_{\min} = \min_{\substack{k,j \\ k \neq j}} d_{k,j}$$

- Since Q(·) is a monotone decreasing function

$$\sum_{\substack{j=1 \\ j \neq k}}^{M} Q\left(\sqrt{\frac{d_{kj}^2}{2N_0}}\right) \leq (M-1)Q\left(\sqrt{\frac{d_{\min}^2}{2N_0}}\right)$$

- Consequently, we may simplify the union bound as

$$\boxed{P_e \leq (M-1)Q\left(\sqrt{\frac{d_{\min}^2}{2N_0}}\right)}$$

Simplified form of union bound

What makes a good signal constellation?